# MODULE 7 : CONDITIONAL PROBABILITY

**SUMMARY Key Concepts**

***Conditional Probability***

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad P(B|A) = \frac{P(B \cap A)}{P(A)}$$

**Note:** $P(A|B)$ is a probability based on the new sample space $B$, while $P(A \cap B)$ and $P(B)$ are probabilities based on the original sample space $S$.

***Product Rule***

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$$

***Independent Events***

- $A$ and $B$ are **independent** if

$$P(A \cap B) = P(A)P(B)$$

- If $A$ and $B$ are independent events with nonzero probabilities, then

$$P(A|B) = P(A) \quad \text{and} \quad P(B|A) = P(B)$$

- If $A$ and $B$ are events with nonzero probabilities and either $P(A|B) = P(A)$ or $P(B|A) = P(B)$, then $A$ and $B$ are independent.
- If $E_1, E_2, \ldots, E_n$ are independent, then

$$P(E_1 \cap E_2 \cap \cdots \cap E_n) = P(E_1)P(E_2) \cdot \cdots \cdot P(E_n)$$

## CONDITIONAL PROBABILITY

Suppose you want to approximate the probability that American Airlines flight 3821 will be delayed on a day that is overcast. To do this you observe AA3821 everyday for n days and record whether or not it was overcast that day, and whether or not the flight was delayed.

| DAY | OVERCAST? B | DELAYED? A |
|---|---|---|
| ~~1~~ | ~~NO~~ | ~~YES~~ |
| 2 | YES ✓ | NO |
| 3 | YES ✓ | NO |
| ~~4~~ | ~~NO~~ | ~~NO~~ |
| 5 | YES ✓ | YES ✓ |
| ~~6~~ | ~~NO~~ | ~~YES~~ |
| ~~7~~ | ~~NO~~ | ~~NO~~ |
| ~~8~~ | ~~NO~~ | ~~NO~~ |
| 9 | YES ✓ | YES ✓ |
| 10 | YES ✓ | NO |
| ⋮ | ⋮ | ⋮ |
| n | YES | YES |

$$P(A) \approx \frac{f_n(A)}{n}$$

$$P(A) = \lim_{n \to \infty} \frac{f_n(A)}{n}$$

IF $n = 10$

$$P(A|B) \approx \frac{f_n(A \cap B)}{f_n(B)} = \frac{2}{5}$$

LET $f_n(A)$ = # TIMES FLIGHT AA 3821 IS DELAYED

$f_n(B)$ = # TIMES THE WEATHER IS OVERCAST

$f_n(A \cap B)$ = # TIMES FLIGHT AA 3821 IS DELAYED AND THE WEATHER IS OVERCAST

$$P(A|B) \approx \frac{f_n(A \cap B)}{f_n(B)} = \frac{f_n(A \cap B)/n}{f_n(B)/n}$$

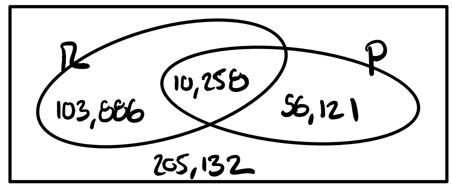
AS $n \to \infty$

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

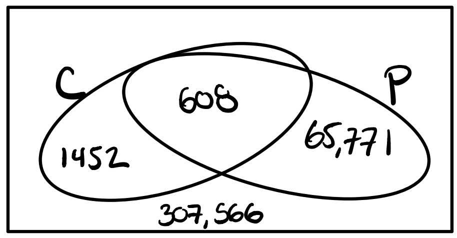
Example. The marketing department for a website that asks visitors to login with their Facebook account has collected data on its users from Facebook Analytics. This data is summarized in the following tables.

1. What percentage of your visitors follow the rock?
2. What percentage of your visitors follow Cats of Instagram?
3. If your company has budgeted \$2500 to spend on Instagram ads, at \$1.25 per click (you only pay when someone clicks the ad/visits your website), should you target your ads toward followers of The Rock or followers of Cats of Instagram?

| Website Visitors That... | Place an order | Do not place an order | Total |
|---|---|---|---|
| Follow The Rock | 10,258 | 103,886 | 114,144 |
| Do not follow The Rock | 56,121 | 205,132 | 261,253 |
| Total | 66,379 | 309,018 | 375,397 |




| Website Visitors That... | Place an order | Do not place an order | Total |
|---|---|---|---|
| Follow Cats of IG | 608 | 1,452 | 2,060 |
| Do not follow Cats of IG | 65,771 | 307,566 | 373,337 |
| Total | 66,379 | 309,018 | 375,397 |




1. $P(R) = \dfrac{114,144}{375,397} = .3041 \rightarrow 30.41\%$

2. $P(C) = \dfrac{2060}{375,397} = .0059 \rightarrow 0.59\%$

3. $\underset{(1)}{P(P \mid R)} \quad \overset{?}{\underset{<}{>}} \quad \underset{(2)}{P(P \mid C)}$

(1) $P(P \mid R) = \dfrac{n(P \cap R)}{n(R)} = \dfrac{10,258}{114,144} = .0899 \rightarrow 8.99\%$

$$\hookrightarrow = \frac{P(P \cap R)}{P(R)} = \frac{10,258 / 375,397}{114,144 / 375,397}$$

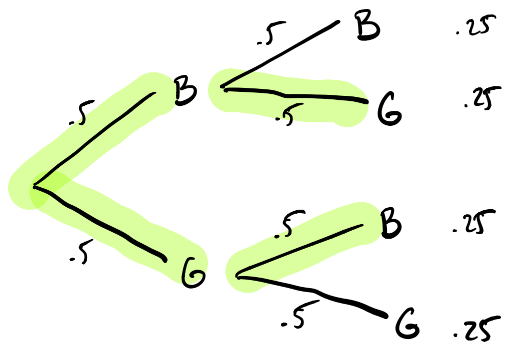
(12) $P(P|C) = \dfrac{P(P \cap C)}{P(C)} = \dfrac{608/375{,}397}{2060/375{,}397} = .2951 \rightarrow 29.51\%$

Example. You know that your neighbors have two children. Then one day you see one of the children playing outside, and this child is a boy. What is the probability that the other child is also a boy?

1B1G 2B 2G

EXP. YOUR NEIGHBOR HAS 2 KIDS, 1st KID, 2nd KID, RECORD B OR G FOR EACH KID.

SAMPLE SPACE $S = \{\underbrace{BB, BG, GB, GG}_{\text{ALL EQUALLY LIKELY}}\}$




$P(BB) = \frac{1}{4}$

$P(BB \mid GG') \rightarrow \frac{1}{3}$

NOTE: $S = \{\underbrace{\overset{1/3}{BB}, \overset{1/3}{BG}, \overset{1/3}{GB}}_{\text{ALL EQUALLY LIKELY}}, \cancel{GG}\}$

# MULTIPLICATION RULE

$$P(B) \times P(A|B) = \frac{P(A\cap B)}{\cancel{P(B)}} \times \cancel{P(B)}$$

$$\boxed{P(A\cap B) = P(B)P(A|B)}$$

CONDITIONAL CLAUSE
⇓
CONDITIONAL PROB.

Example. Your company is about to begin a kickstarter campaign and wants to know the probability that you will be able to raise 1 million dollars in 8 weeks. You know that if the campaign video goes viral then the probability is 95%, and if the campaign does not go viral then there is a 70% chance. Suppose there is a 15% chance that the campaign video will go viral.

1. What is the probability that the campaign video goes viral but the campaign is not able to raise 1 million dollars?
2. What is the probability that the campaign does not go viral and the campaign still raises 1 million dollars?
3. What is the probability that the campaign raises 1 million dollars?

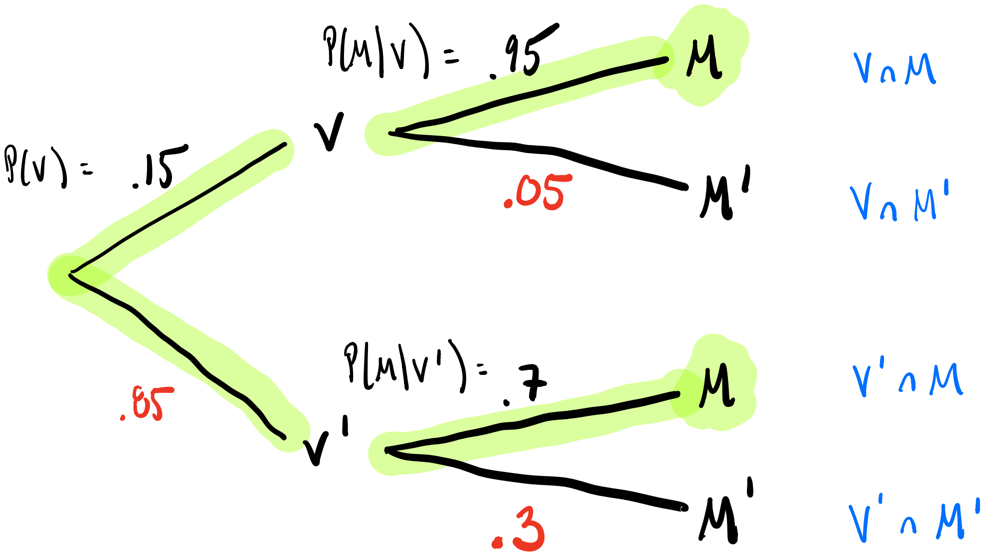
NOTATION: $V$ = VIDEO GOES VIRAL

$M$ = RAISE 1 MILLION DOLLARS

GIVEN: $P(M|V) = .95$

$P(M|V') = .70$

$P(V) = .15$



1. $P(V \cap M') = P(V)P(M'|V) = (.15)(.05) = .0075$

2. $P(V' \cap M) = P(V')P(M|V') = (.85)(.7) = .595$

3. $P(M) = P((V \cap M) \cup (V' \cap M))$

$$= P(V \cap M) + P(V' \cap M) - \underbrace{P(\overbrace{(V \cap M) \cap (V' \cap M)}^{\varnothing})}_{0}$$

$$= P(V)P(M|V) + P(V')P(M|V')$$

$$= (.15)(.95) + (.85)(.7) = .7375$$

## Independent Events

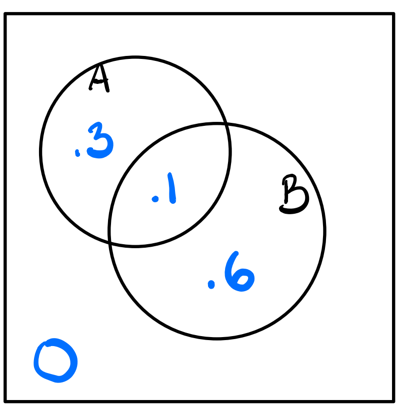
IF $P(A|B) = P(A)$ THEN $A$ & $B$ ARE INDEPENDENT EVENTS

IF $P(B|A) = P(B)$ THEN $A$ & $B$ ARE INDEPENDENT EVENTS

e.g.

$P(A) = .4$ $P(B) = .7$

| | A | A' | |
|---|---|---|---|
| B | .1 | .6 | .7 |
| B' | .3 | 0 | .3 |
| | .4 | .6 | |




$P(A) = .4$ $\quad P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.1}{.7} = .14$

$P(A) \neq P(A|B)$ A, B NOT INDEPENDENT

| | A | A' | |
|---|---|---|---|
| B | .28 | .42 | .7 |
| B' | .12 | .18 | .3 |
| | .4 | .6 | |

$$P(A) = .4 \qquad P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{.28}{.7} = .4$$

A & B ARE INDEPENDENT!

**74.** Ann and Barbara are playing a tennis match. The first player to win 2 sets wins the match. For any given set, the probability that Ann wins that set is $\frac{2}{3}$. Find the probability that

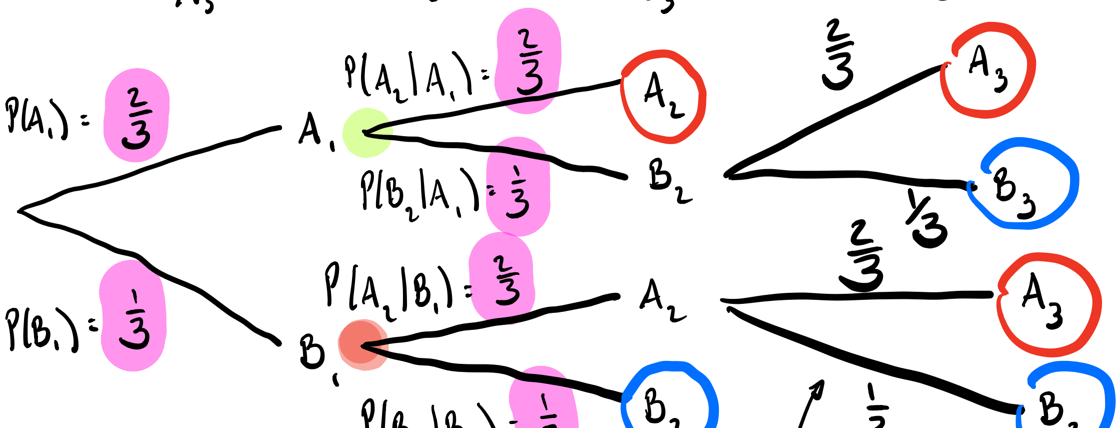
(A) Ann wins the match.

(B) 3 sets are played.

(C) The player who wins the first set goes on to win the match.

LET $A_1$ = ANN WINS 1st SET　　$B_1$ = BARBARA WINS 1st SET
$A_2$ = 2nd　　$B_2$ = 2nd
$A_3$ = 3rd　　$B_3$ = 3rd

$A_1, A_2, A_3,$
$B_1, B_2, B_3$
ARE ALL INDEPENDENT



$P(B_2|B_1) = 3$ 2 3 3

$P(B_3 \mid B_1 \cap A_2)$

$$P(\text{ANN WINS}) = P(A_1 \cap A_2) + P(A_1 \cap B_2 \cap A_3) + P(B_1 \cap A_2 \cap A_3)$$

$$= P(A_1)P(A_2 \cancel{| A_1}) + P(A_1)P(B_2 \cancel{| A_1})P(A_3 \cancel{| A_1 \cap B_2})$$

$$+ P(B_1)P(A_2 \cancel{| B_1})P(A_3 \cancel{| B_1 \cap A_2})$$

SINCE ALL EVENTS ARE INDEPENDENT

$$= P(A)P(A) + P(A)P(B)P(A) + P(B)P(A)P(A)$$

$$= \left(\frac{2}{3}\right)^2 + 2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)$$

$$= .7407$$

# Bayes Formula

**THEOREM 1 Bayes' Formula**

Let $U_1, U_2, \ldots, U_n$ be $n$ mutually exclusive events whose union is the sample space $S$. Let $E$ be an arbitrary event in $S$ such that $P(E) \neq 0$. Then,

$$P(U_1|E) = \frac{P(U_1 \cap E)}{P(E)} = \frac{P(U_1 \cap E)}{P(U_1 \cap E) + P(U_2 \cap E) + \cdots + P(U_n \cap E)}$$

$$= \frac{P(E|U_1)P(U_1)}{P(E|U_1)P(U_1) + P(E|U_2)P(U_2) + \cdots + P(E|U_n)P(U_n)}$$

Similar results hold for $U_2, U_3, \ldots, U_n$.

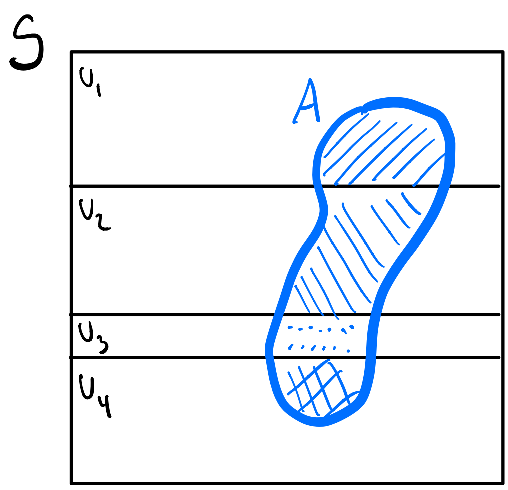
LAW OF TOTAL PROBABILITY:

SUPPOSE EVENTS $U_1, U_2, \ldots, U_n$ SATISFYING

(1) $U_1 \cup U_2 \cup \ldots \cup U_n = S$ (EVERYTHING)

(2) $U_i \cap U_j = \emptyset$, $i \neq j$.

THEN FOR ANY EVENT $A \subseteq S$ WE HAVE




$$A = (A \cap U_1) \cup (A \cap U_2) \cup \ldots \cup (A \cap U_n)$$

$$P(A) = P(A \cap U_1) + P(A \cap U_2) + \ldots + P(A \cap U_n)$$

$\Big\downarrow$ PRODUCT RULE

$$P(A) = P(U_1)P(A|U_1) + P(U_2)P(A|U_2) + \ldots + P(U_n)P(A|U_n)$$

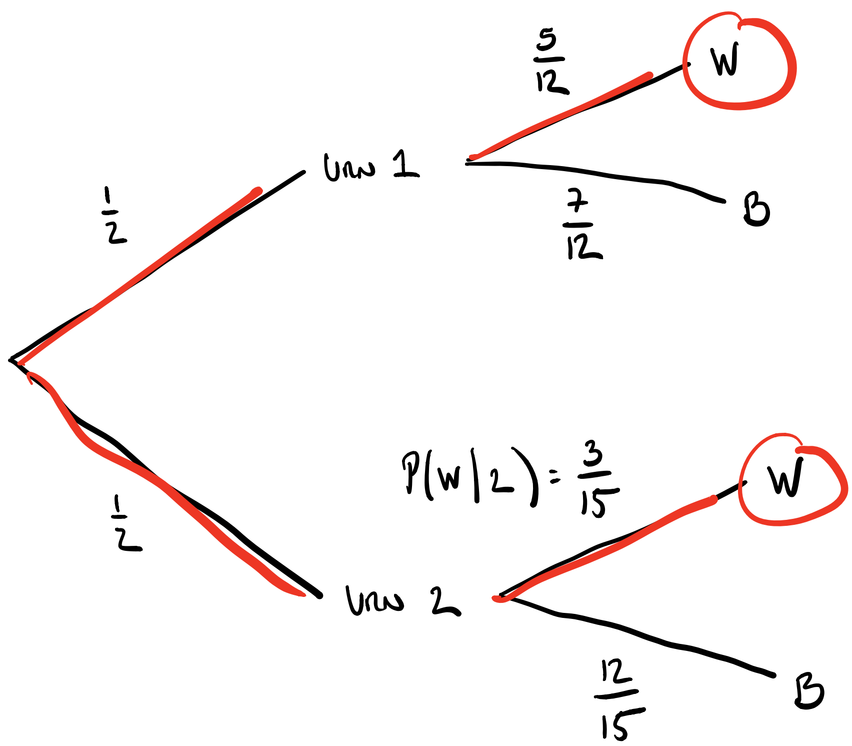
## Bayes' Formula:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B|A)}{P(B)}$$

Bayes' Formula

## LoTP + BF combined:

Urn 1 contains 5 white balls and 7 black balls. Urn 2 contains 3 whites and 12 black. A fair coin is flipped; if it is Heads, a ball is drawn from Urn 1, and if it is Tails, a ball is drawn from Urn 2. Suppose that this experiment is done and you learn that a white ball was selected. What is the probability that this ball was in fact taken from Urn 2? (i.e., that the coin flip was Tails)




Find $$P(2|W) \overset{BF}{=} \frac{P(2)P(W|2)}{P(W)} \overset{L.o.T.P.}{=} \frac{P(2)P(W|2)}{P(1)P(W|1) + P(2)P(W|2)}$$

$$= \frac{\left(\frac{1}{2}\right)\left(\frac{3}{15}\right)}{\left(\frac{1}{2}\right)\left(\frac{5}{12}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{15}\right)} = \boxed{.3243}$$