§7.2 Sets

A set is a collection of objects, called elements, such that is always possible to test whether a given object is in the set or not. One way of defining a set is by listing its elements inside curly brackets. The order in which the laments are listed does not matter.

$$S = \{a, b, c, d, e, f\}$$

SET

SPECIAL SETS:

NATURAL #'S $\mathbb{N} = \{1, 2, 3, \dots\}$

INTEGERS $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

RATIONAL #'S $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\}$

FRACTIONS

NUMERATOR p

DENOMINATOR q

SUCH THAT

p & q BELONG TO

THE SET OF INTEGERS

SET BUILDER NOTATION

EX. DESCRIBE EACH SET IN WORDS.

(a) $A = \left\{ \frac{m}{n^2} \mid m, n \in \mathbb{Z}, n \neq 0 \right\}$

$0 \in A$

$\frac{-2}{9} = \frac{-2}{3^2}$ $\quad m = -2 \in \mathbb{Z}$, $n = 3 \in \mathbb{Z}$ $\Rightarrow \frac{-2}{9} \in A$

$\frac{3}{5} = \frac{3}{\sqrt{5}^2}$ $\quad 3 \in \mathbb{Z}$, $\sqrt{5} \notin \mathbb{Z}$ $\Rightarrow \frac{3}{5} \notin A$

A IS THE SET OF ALL FRACTIONS (RATIONAL #'s) WITH DENOM = PERFECT SQUARE.

(b.) $B = \left\{ \frac{r}{2^s} \mid r, s \in \mathbb{N} \right\}$

$0 \notin B$

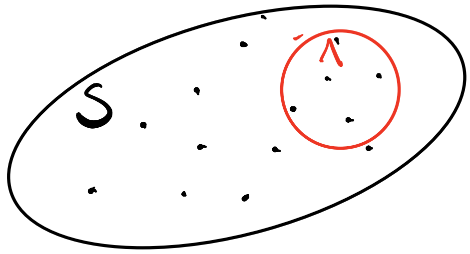
POS. FRACTIONS W/ DENOM = POWER OF 2

Ex. There is a village where the barber cuts the hair of everyone who does not cut their own hair, and he only cuts the hair of people who do not cut their own hair. Consider the set of people whose hair the barber cuts. Is the barber in this set? That is, who cuts the barber's hair?

Ex. Consider the set of all sets that do not contain themselves.

Both lead to paradoxes.

A set A is a subset of a set S



$$A \subseteq S$$

if every element of A is an element of S. Note that the element x and the set containing it {x} are two different types of objects. Similarly, the empty set

$$\emptyset \quad \text{or} \quad \{\ \}$$

and the set containing the empty set

$$\{\emptyset\} \quad \text{or} \quad \{\{\ \}\}$$

are two different types of objects.

Ex. Let W be the following set.
W={Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}
How many distinct subsets of W exist?

SUBSET $A \subseteq W$

MON	TUE	WED	THU	FRI	SAT	SUN
$\downarrow$ Y/N	$\downarrow$ Y/N	$\downarrow$ Y/N	$\downarrow$ Y/N	$\downarrow$ Y/N	$\downarrow$ Y/N	$\downarrow$ Y/N

$A = \{$ MON, THU, FRI, SUN $\}$

\# POSSIBLE SUBSETS $= \boxed{2^7 = 128}$

The set of all sunsets of a given set W is called the power set of W.

NOTE: $\emptyset \subseteq W$ FOR ALL SETS W.

$W \subseteq W$ FOR ALL SETS W

IF $A \subseteq S$ AND $A \neq S$,

THEN $A \subset S$ IT IS A PROPER SUBSET

<u>ex.</u> LET $S = \{ \emptyset, \{\emptyset\} \}$

LIST ALL DISTINCT SUBSETS OF S.

$4 = 2^2$ POSSIBLE SUBSETS

$\emptyset$

$\{\emptyset\}$

$\{\{\emptyset\}\}$

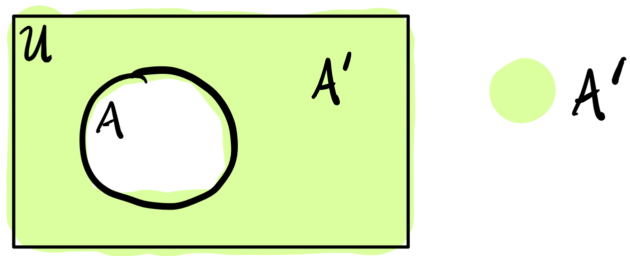
$\{\emptyset, \{\emptyset\}\}$

<u>Def</u>: THE UNIVERSAL SET $\mathcal{U}$ IS THE SET OF ALL ELEMENTS UNDER CONSIDERATION.

GIVEN A SET $A \subseteq \mathcal{U}$, WE DEFINE

THE COMPLIMENT

$$A' = \{ x \in \mathcal{U} \mid x \notin A \}$$

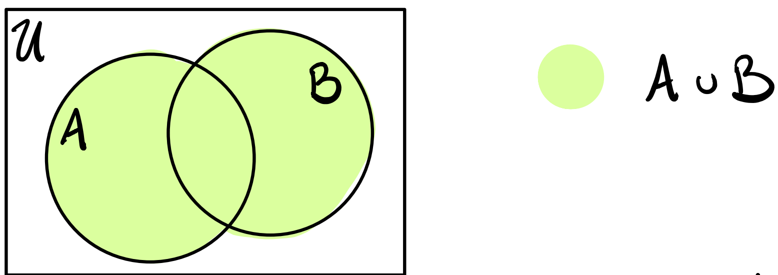


GIVEN TWO SETS $A, B \subseteq \mathcal{U}$, WE DEFINE

THE **UNION**

$$A \cup B = \{ x \in \mathcal{U} \mid \underbrace{x \in A \text{ or } x \in B}_{\text{or both! (at least one)}} \}$$

CUP

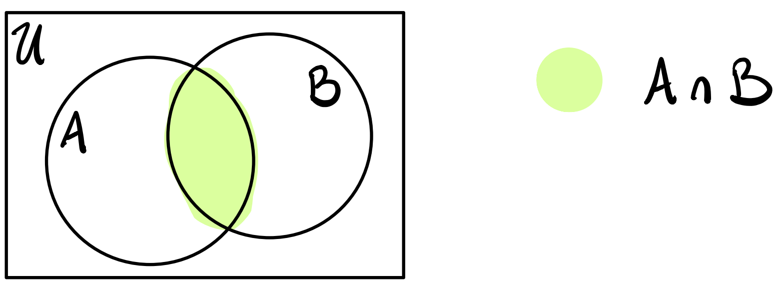


n!

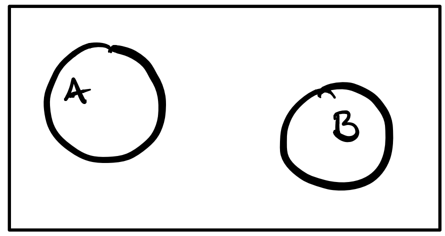
WE DEFINE THE **INTERSECTION**

$$A \cap B = \{ x \in \mathcal{U} \mid x \in A \text{ and } x \in B \}$$

CAP



SPECIAL CASE: $\underbrace{A \cap B = \emptyset}$



WE SAY A & B ARE MUTUALLY EXCLUSIVE

(DISJOINT)

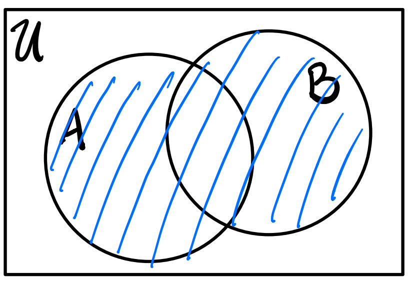
"NO OVERLAP"

ex. USE VENN DIAGRAMS TO PROVE DE MORGAN'S LAW:

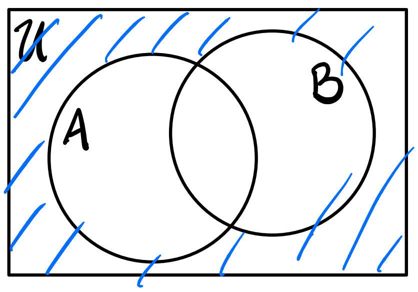
(a) $\boxed{(A \cup B)'} = \boxed{A' \cap B'}$

(b) $(A \cap B)' = A' \cup B'$

LEFT

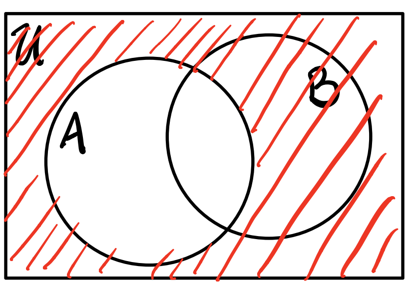


$A \cup B$

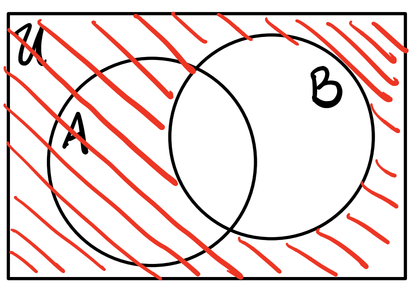


$(A \cup B)'$

RIGHT

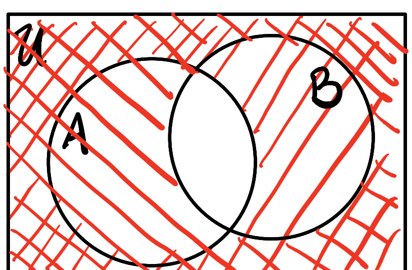


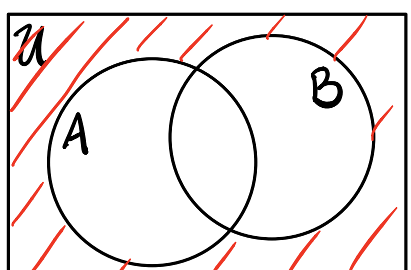
A'



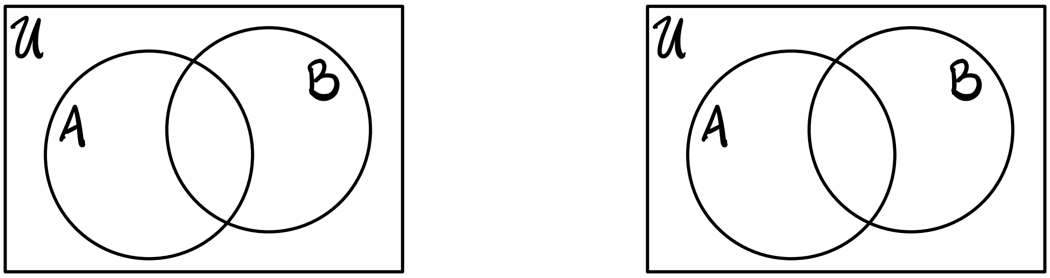
B'

SAME





$A' \cap B'$ $\qquad$ $A' \cap B'$



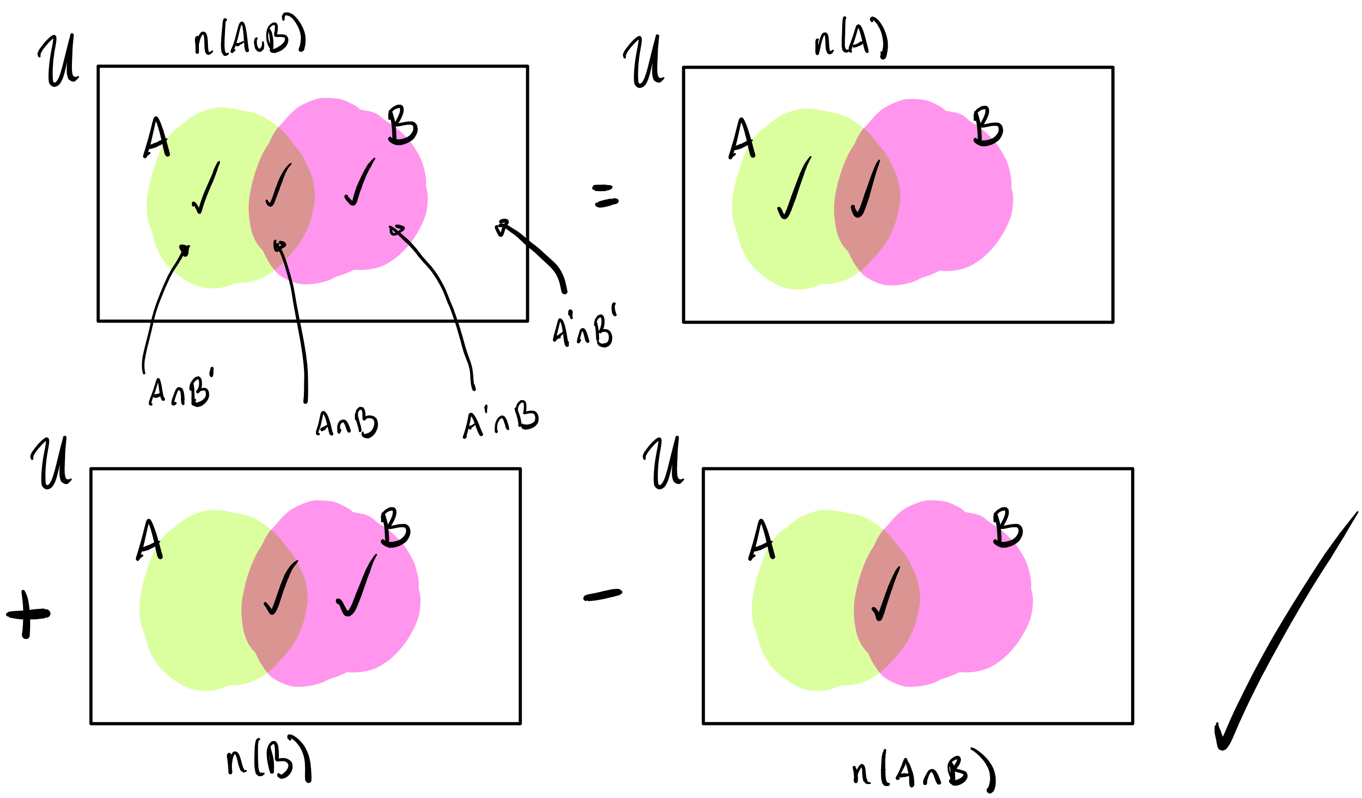
Def: Given a set A, let

$$n(A) = \#\text{ elements in } A.$$

The Addition Rule

ex. Use Venn diagrams to show

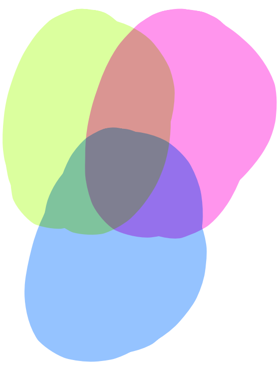
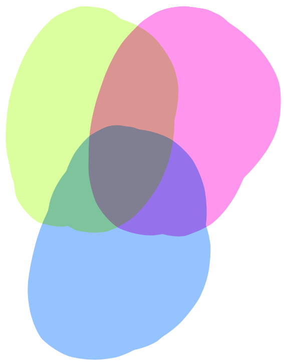
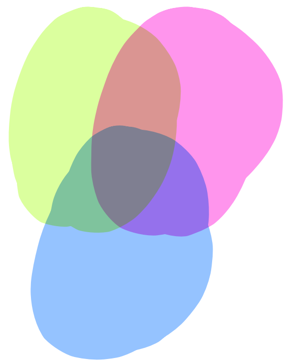
(a) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$



(b) $n(A\cup B\cup C) = n(A)+n(B)+n(C)$

$-\,n(A\cap B) - n(B\cap C) - n(C\cap A)$

$+\,n(A\cap B\cap C)$

 $-$  $+$ 

$$\underbrace{n(A)+n(B)+n(C)}_{\text{INTERSECTIONS OF } 1}\ \underbrace{-n(A\cap B)-n(B\cap C)-n(C\cap A)}_{\text{INTERSECTIONS OF } 2}\ \underbrace{+\,n(A\cap B\cap C)}_{\text{INTERSECTIONS OF } 3}$$

ALTERNATING SUM

§8.1 Sample Space, Events, Probability

An experiment is any procedure by which an observation is made. The set of all possible observations/ outcomes of an experiment is called the sample space S — it is the universal set. A subset A of the sample space S is called an event.

$n(S) = {}_{52}C_2 = \frac{52!}{2!\,50!}$

$\frac{52 \cdot 51}{2} = 1326$

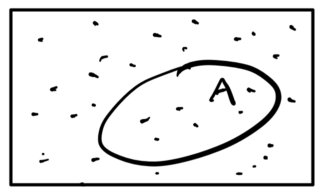
e.g.
Experiment: select two cards from a deck of 52 cards.
Sample space: All possible combinations of 2 cards taken from 52.
A Event: selecting two cards with the same face value (a "pair").

$n(A) = ?$

Now we introduce a theoretical way to measure the likelihood of an event.

Imagine repeating the experiment n times, and each time you record whether or not the event A is observed or not.

S



Repetition	Was event A observed?
1	No
2	No
3	Yes
4	No
5	Yes
⋮	⋮
n	No

Yes

↓

Let $f_n(A)$ be the # times A is observed in n repetitions of the experiment

$f_n(A)$ IS THE FREQUENCY WITH WHICH EVENT A IS OBSERVED IN n REPITITIONS.

e.g. $f_n(A) = 21$, $n = 100$ $\rightsquigarrow$ $\text{Prob}(A) \approx \frac{21}{100}$

INTUITIVELY, THE LIKELIHOOD OF A IS APPROXIMATELY

$$\frac{f_n(A)}{n} \quad \begin{array}{l} \leftarrow \text{FREQUENCY } A \text{ OCCURRED} \\ \leftarrow \text{TOTAL NUMBER OF REPITITIONS} \end{array}$$

AND THIS APPROXIMATION GETS BETTER AS n GETS LARGER.*

THUS, WE DEFINE THE PROBABILITY OF A

$$P(A) = \lim_{n \to \infty} \frac{f_n(A)}{n}$$

$$\frac{0}{n} \le \underbrace{\frac{f_n(A)}{n}}_{P(A)} \le \frac{n}{n}$$

NOTE: $\underset{\text{NEVER}}{0} \le P(A) \le \underset{\text{ALWAYS}}{1}$

SPECIAL CASE:

(1) SAMPLE SPACE $S = \{e_1, e_2, \ldots, e_n\}$

IS FINITE

(2) ALL POSSIBLE OUTCOMES ARE EQUALLY LIKELY.

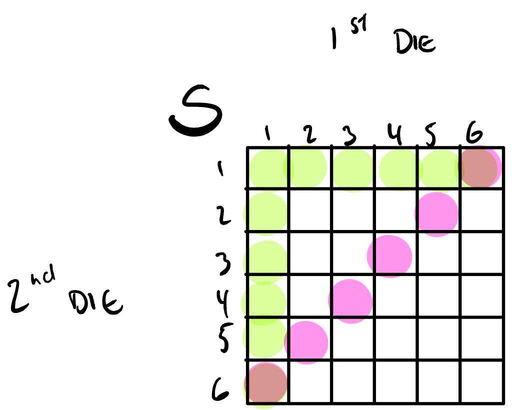
IN THIS CASE,

$$P(A) = \frac{n(A)}{n(S)}$$

WAYS THAT EVENT A CAN HAPPEN

TOTAL # OF POSSIBLE OUTCOMES.

Ex. An experiment is performed by rolling two dice. What is the probability that at least one of the dice shows a 1 or the total of the two dice is a 7?



EACH □ REPRESENT A POSSIBLE OUTCOME

$n(S) = 36$

A = EVENT OF ROLLING ≥ 1 ONE

B = EVENT OF ROLLING A TOTAL 7

$n(A) = 11 \quad n(B) = 6 \quad n(A \cap B) = 2$

FIND $P(A \cup B) = \dfrac{n(A \cup B)}{n(S)}$

$$= \frac{n(A) + n(B) - n(A \cap B)}{n(S)} = \frac{11 + 6 - 2}{36}$$

$$= \frac{15}{36} = \boxed{\frac{5}{12}}$$

§ 8.2 UNIONS, INTERSECTIONS, AND COMPLEMENTS OF EVENTS

INCLUSION/EXCLUSION PRINCIPAL

$$P(A \cup B) = \frac{n(A \cup B)}{n(S)}$$

$$= \frac{n(A) + n(B) - n(A \cap B)}{n(S)}$$

$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$

$$\underbrace{P(A \cup B)}_{\text{UNION}} = P(A) + P(B) - \underbrace{P(A \cap B)}_{\text{INTERSECTION}}$$

$$\underbrace{P(A \cap B)}_{\text{INTERSECTION}} = P(A) + P(B) - \underbrace{P(A \cup B)}_{\text{UNION}}$$

Ex. If you flip a coin 20 times, what is the probability that you flip exactly 8 heads?

Ex. Twelve friends have gotten together to watch a football game. 5 are rooting for team A and 7 are rooting for team B. If three friends are chosen randomly to go buy pizza, what is the probability that they all root for the same team? What is the probability that two friends root for one team and one friend roots for the other.

PROBABILITIES OF COMPLIMENTARY EVENTS

$$P(A') = \frac{n(A')}{n(S)}$$

$$= \frac{n(S) - n(A)}{n(S)}$$

$$= \frac{n(S)}{n(S)} - \frac{n(A)}{n(S)}$$

$$P(A') = 1 - P(A)$$

$$P(A) + P(A') = 1$$