MODULE 5: COUNTING PRINCIPALS

THEOREM 2 Multiplication Principle (for Counting)

1. If two operations O_1 and O_2 are performed in order, with N_1 possible outcomes for the first operation and N_2 possible outcomes for the second operation, then there are

$$N_1 \cdot N_2$$

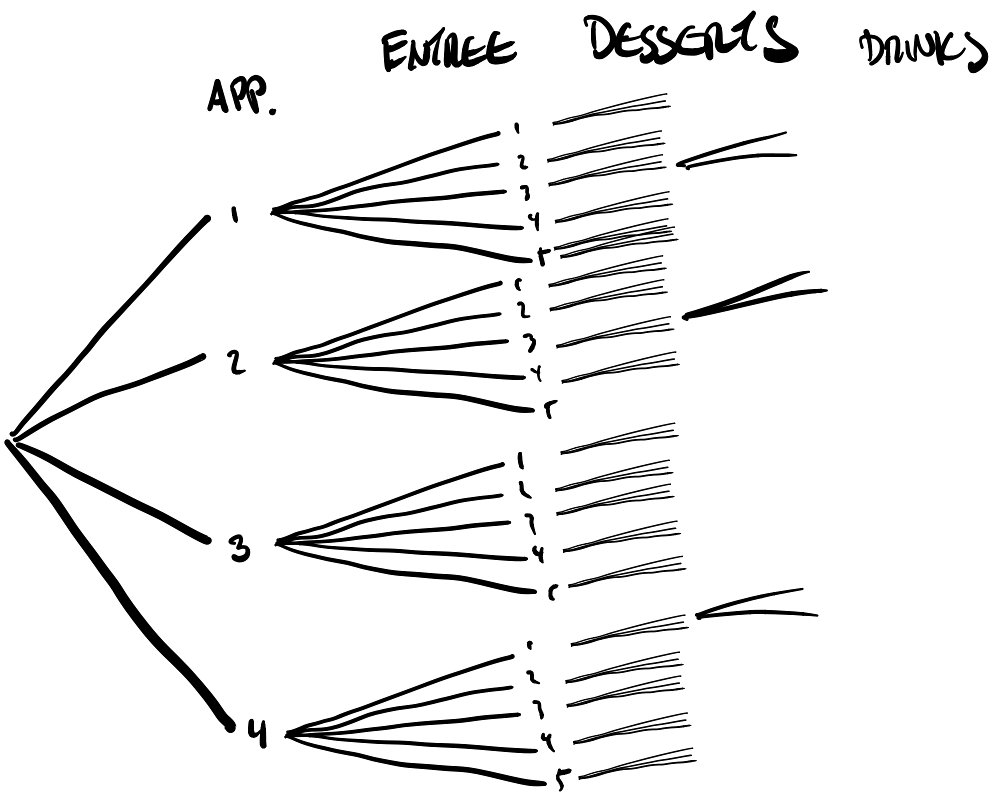
possible combined outcomes of the first operation followed by the second.

2. In general, if n operations $O_1, O_2, \ldots, O_n$ are performed in order, with possible number of outcomes $N_1, N_2, \ldots, N_n$, respectively, then there are

$$N_1 \cdot N_2 \cdot \cdots \cdot N_n$$

possible combined outcomes of the operations performed in the given order.

Ex. If a restaurant serves 4 appetizers, 5 entrees, 3 desserts, and 2 drinks, how many meals can be purchased from this restaurant? A meal consists of one app, one entree, 1 dessert, and 1 drink.



4 GROUPS OF 5 GROUPS OF 3 GROUPS OF 2

$= 4 \times 5 \times 3 \times 2 = 120$

Example 1.

Suppose a company requires its customers to create a PIN composed of 4 digits 0-9.

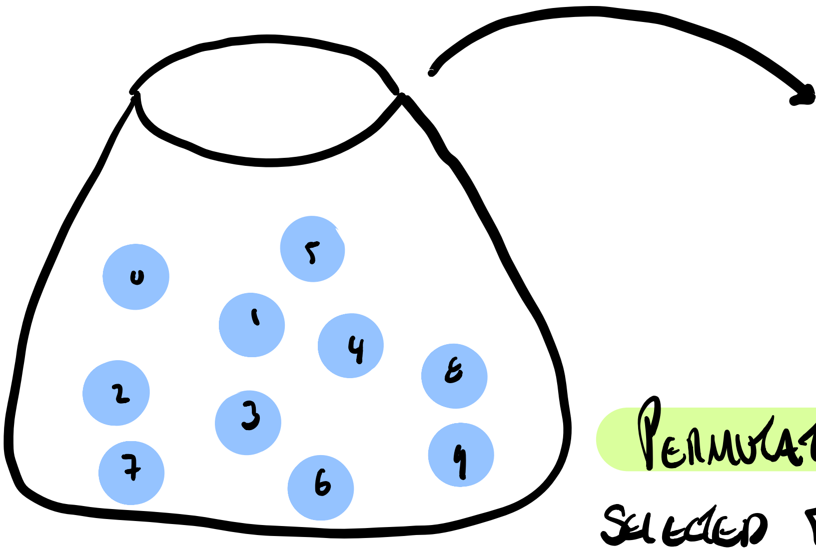
1. How many PINs are possible?
2. How many PINs are possible if successive digits must be different?
3. How many PINs are possible if each digit must be distinct?

\# POSSIBLE OUTCOMES FOR EACH

1. $$\underset{1^{st}}{\underline{10}} \times \underset{2^{nd}}{\underline{10}} \times \underset{3^{rd}}{\underline{10}} \times \underset{4^{th}}{\underline{10}} = 10{,}000$$

2. $$\underset{1^{st}}{\underline{10}} \times \underset{2^{nd}}{\underline{9}} \times \underset{3^{rd}}{\underline{9}} \times \underset{4^{th}}{\underline{9}} = 7{,}200$$

3. $$\underset{1^{st}}{\underline{10}} \times \underset{2^{nd}}{\underline{9}} \times \underset{3^{rd}}{\underline{8}} \times \underset{4^{th}}{\underline{7}} = 5{,}040 = {}_{10}P_4$$



SELECT 4 OBJECTS FROM 10 OBJECTS, ORDER OF SELECTION MATTERS.

PERMUTATION OF $r = 4$ OBJECTS SELECTED FROM $n = 10$ OBJECTS

DEFINITION Permutation of *n* Objects Taken *r* at a Time

A permutation of a set of *n* distinct objects taken *r* at a time without repetition is an arrangement of *r* of the *n* objects in a specific order.

THEOREM 2 Number of Permutations of *n* Objects Taken *r* at a Time

The number of permutations of n distinct objects taken r at a time without repetition is given by*

$${}_nP_r = n(n-1)(n-2)\cdot\cdots\cdot(n-r+1) \qquad r \text{ factors}$$

$${}_5P_2 = 5\cdot 4 \text{ factors}$$

or

$${}_nP_r = \frac{n!}{(n-r)!} \qquad 0 \le r \le n \qquad {}_5P_2 = \frac{5!}{(5-2)!} = \frac{5!}{3!}$$

Note: ${}_nP_n = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!} = n!$ permutations of n objects taken n at a time.

Remember, by definition, $0! = 1$.

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* In place of the symbol ${}_nP_r$, the symbols P_r^n, $P_{n,r}$, and $P(n, r)$ are often used.

$n!$ "N FACTORIAL"

$n! = n(n-1)(n-2)\cdots(2)(1)$

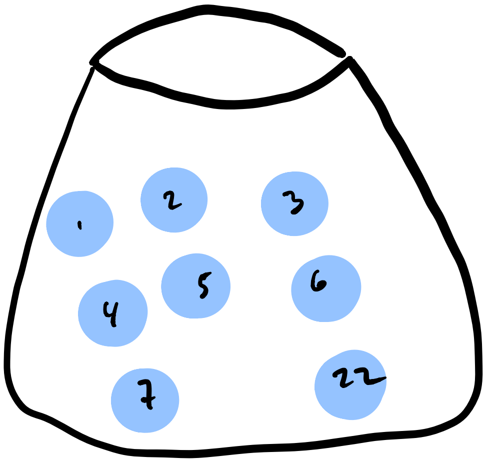
NOTE: # WAYS TO ARRANGE ALL N OBJECTS (FROM N OBJECTS)
IS ${}_nP_n = n(n-1)(n-2)\cdots(2)(1) = n!$

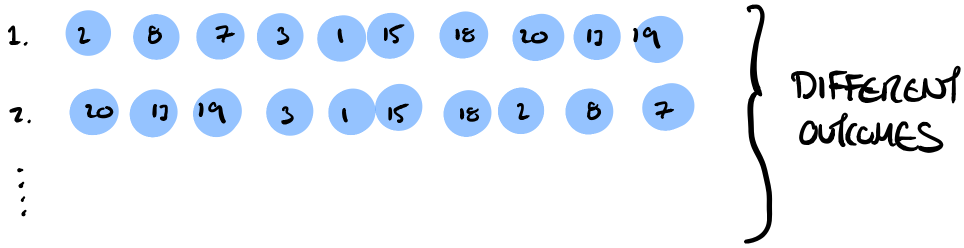
THE FORMULA WOULD BE ${}_nP_n = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!}$

BY CONVENTION $0! = 1$

Example 2.

A musician has written and recorded 22 songs. If she want to release 10 of these songs as an album (order matters!), how many possible ways are there for her to do this?





$$ {}_{22}P_{10} = \frac{22!}{(22-10)!} = \frac{22!}{12!} $$

$$ = \frac{22 \times 21 \times 20 \times \dots \times 13 \times \cancel{12} \times \cancel{11} \times \cancel{10} \times \dots \times \cancel{2} \times \cancel{1}}{\cancel{12} \times \cancel{11} \times \cancel{10} \times \dots \times \cancel{2} \times \cancel{1}} $$

$$ = \underline{22} \times \underline{21} \times \underline{20} \times \underline{19} \times \underline{18} \times \underline{17} \times \underline{16} \times \underline{15} \times \underline{14} \times \underline{13} $$

$$ = 2.3465 \times 10^{12} $$

Example 3.

A musician has written a song with the following structure:

Verse, Chorus, Verse, Chorus, Bridge, Verse, Chorus.

Now a lyricists has written 5 different verses, 2 different choruses, and 3 different bridges. If each verse must be different, and each chorus must be the same, how many songs can be created from these lyrics with the given structure?

BREAK INTO STAGES (3)

$$ \underbrace{\underline{5}\ \underline{4}\ \underline{3} = {}_5P_3}_{\text{CHOOSE VERSES } 1^{st},\ 2^{nd},\ 3^{rd}} \times \underbrace{2 = {}_2P_1}_{\text{CHOOSE A CHORUS}} \times \underbrace{3 = {}_3P_1}_{\text{CHOOSE A BRIDGE}} = \boxed{360} $$

DEFINITION **Combination of *n* Objects Taken *r* at a Time**

A **combination** of a set of n distinct objects taken r at a time without repetition is an r-element subset of the set of n objects. The arrangement of the elements in the subset does not matter.

Example 4.

How many ways are there to choose 3 aces from 4 aces if the order doesn't matter?

Let S, C, H, and D represent the ace of spades, ace of clubs, ace of diamonds, and ace of hearts, respectively.

DIFFERENT PERMUTATIONS, BUT THE SAME COMBINATION

4 DIFF. COMBINATIONS

SCH, SHC, CSH, CHS, HSC, HCS,
CHD, CDH, HCD, HDC, DCH, DHC,
HDS, HSD, DHS, DSH, SHD, SDH,
DSC, DCS, SDC, SCD, CDS, CSD.

$$_3P_3 = 3! = 6$$

\# OF COMBINATIONS OF 3 OBJECTS FROM 4 OBJECTS

IS $$_4C_3 = \frac{_4P_3}{3!} = \frac{24}{6} = \boxed{4}$$

THEOREM 3 Number of Combinations of *n* Objects Taken *r* at a Time

The number of combinations of n distinct objects taken r at a time without repetition is given by*

$$
\begin{aligned}
{}_nC_r &= \binom{n}{r} \\
&= \frac{{}_nP_r}{r!} \\
&= \frac{n!}{r!(n-r)!} \qquad 0 \le r \le n
\end{aligned}
\qquad\qquad
\begin{aligned}
{}_{52}C_5 &= \binom{52}{5} \\
&= \frac{{}_{52}P_5}{5!} \\
&= \frac{52!}{5!(52-5)!}
\end{aligned}
$$

* In place of the symbols ${}_nC_r$ and $\binom{n}{r}$, the symbols C_r^n, $C_{n,r}$ and $C(n, r)$ are often used.

Example 7.

The fish section of a pet store is stocked with 8 guppies, 6 angelfish, 13 goldfish, and 9 rainbowfish.

1. How many ways are there to select one of each?
2. How many ways are there to select two of each?
3. How many ways are there to select 7 fish, if you must select at least one of each?

1. $\underset{GU}{\underline{8}} \times \underset{A}{\underline{6}} \times \underset{GO}{\underline{13}} \times \underset{R}{\underline{9}} = 5616.$

2. $\underset{\text{2 out of 8 Guppies}}{\underline{{}_8C_2}} \times \underset{\text{2 out of 6 Ang. Fish.}}{\underline{{}_6C_2}} \times \underset{\text{2 out of 13 Gold F.}}{\underline{{}_{13}C_2}} \times \underset{\text{2 out of 9 Rainbow F.}}{\underline{{}_9C_2}}$

$$28 \times 15 \times 78 \times 36 = \boxed{1,179,360}$$

$$\frac{13!}{2!\,11!} = \frac{13 \cdot 12}{2} \qquad \frac{9 \cdot 8}{2}$$

3. 7 FISH, AT LEAST ONE OF EACH TYPE.

(8 GU, 6 AF, 13 GF, 9 RF)

$$\underbrace{{}_{8}C_{1}}_{\text{CHOOSE 1 GUPPY}} \times \underbrace{{}_{6}C_{1}}_{\text{CHOOSE 1 AF}} \times \underbrace{{}_{13}C_{1}}_{\text{CHOOSE 1 GF}} \times \underbrace{{}_{9}C_{1}}_{\text{CHOOSE 1 RF}} \times \underbrace{{}_{32}C_{3}}_{\text{CHOOSE 3 FISH OUT OF 32 REMAINING FISH.}}$$

$$8 \times 6 \times 13 \times 9 \times \underbrace{\frac{32 \cdot 31 \cdot \cancel{30}^{5}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}}}_{4960} = \boxed{27,855,360}$$

Example 5.

If you flip a coin 20 times, you get a sequence of heads (H) and tails (T).

1. How many different sequences of heads and tails are possible?
2. How many different sequences of heads and tails have exactly 8 heads?

H T H H T H T T H T H H H H H T T H T T

1. 20 OPERATIONS - EACH WITH 2 POSSIBLE OUTCOMES:

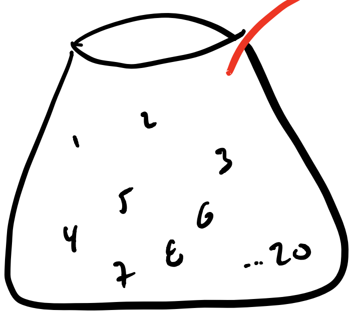
$$2 \times 2 \times 2 \times \ldots \times 2 = 2^{20} = \boxed{1,048,576}$$

2.

H	T	H	T	T	T	H	T	T	T	H	T	T	T	H	H	H	T	H	T
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

(1) CHOOSE WHICH 8 OF THE 20 POSITIONS IN THE SEQUENCE WILL BE HEADS. # WAYS TO DO THIS : $_{20}C_8 = \boxed{125,970}$

(2) ALL THE REST BECOME TAILS. # WAYS TO DO THIS : 1

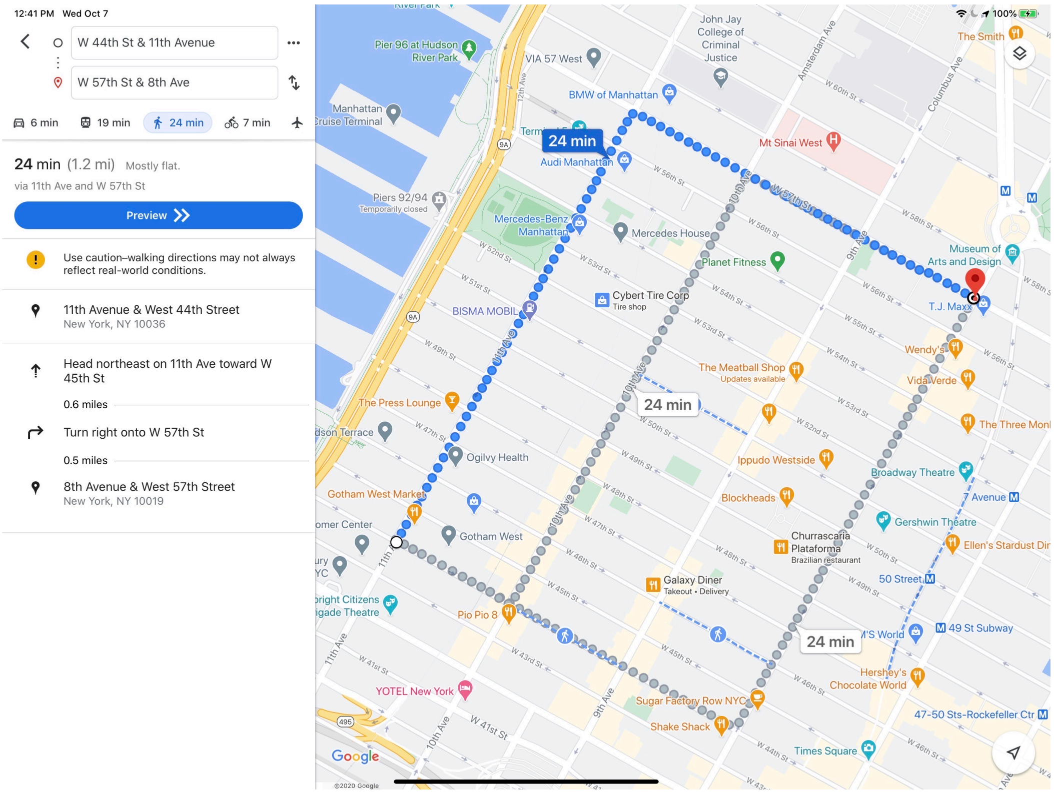


$$\left.\begin{matrix} 1 & 3 & 7 & 11 & 15 & 16 & 17 & 19 \\ 3 & 11 & 15 & 1 & 7 & 17 & 16 & 19 \end{matrix}\right\}$$ RESULT IN H IN SAME POSITIONS

$\Rightarrow$ SAME SEQ

CHALLENGE #1:

Example 6.
How many ways are there to walk from corner of W 44 St and 11th Ave to the corner of W 57 St and 8th Ave, assuming you do not go out of your way (only walk north and east)?



(HINT: THIS IS SIMILAR TO EX. 5.)

CHALLENGE #2

Example 9.
Suppose you want to give each of your 4 nieces/nephews a Christmas card containing some money. You have ten (identical) $20 bills that you want to distribute between the 4 (distinct) envelopes so that no envelope contains $0. How many ways are there to do this?
https://en.wikipedia.org/wiki/Stars_and_bars_(combinatorics)