THEOREM 2 Multiplication Principle (for Counting)

1. If two operations O_1 and O_2 are performed in order, with N_1 possible outcomes for the first operation and N_2 possible outcomes for the second operation, then there are

$$N_1 \cdot N_2$$

possible combined outcomes of the first operation followed by the second.

2. In general, if n operations $O_1, O_2, \ldots, O_n$ are performed in order, with possible number of outcomes $N_1, N_2, \ldots, N_n$, respectively, then there are

$$N_1 \cdot N_2 \cdot \cdots \cdot N_n$$

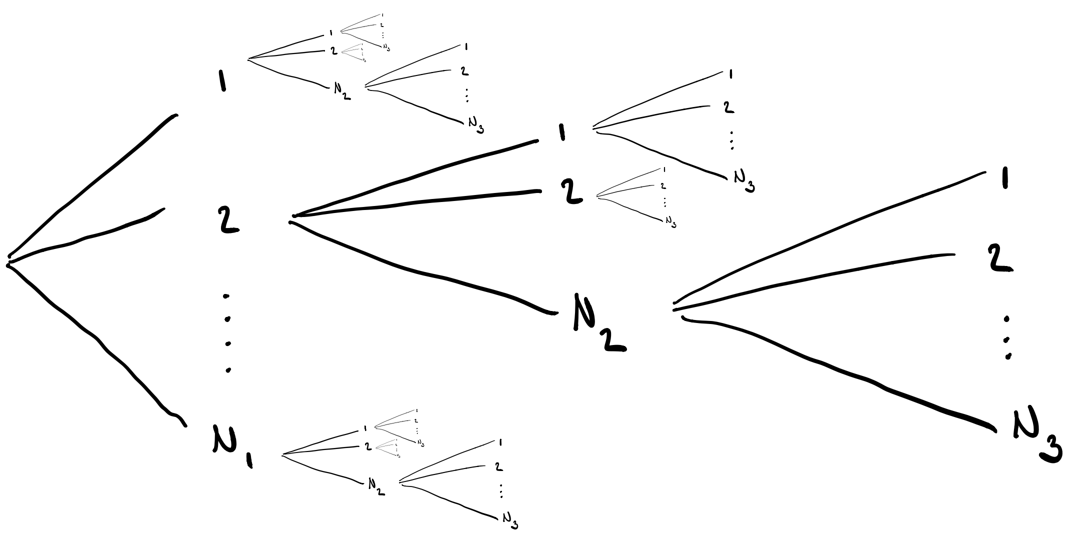
possible combined outcomes of the operations performed in the given order.

COUNT THE # OF POSSIBILITIES...

PATH FROM LEFT TO RIGHT THROUGH DECISION TREE

BREAK INTO A SEQUENCE OF DECISIONS:

$$\underset{\text{\# CHOICES}}{\underline{\;N_1\;}} \times \underset{\text{\# CHOICES}}{\underline{\;N_2\;}} \times \underset{\text{\# CHOICES}}{\underline{\;N_3\;}} = \text{TOTAL \# POSSIBILITIES}$$



Example 1.

Suppose a company requires its customers to create a PIN composed of 4 digits 0-9.

1. How many PINs are possible?
2. How many PINs are possible if successive digits must be different?
3. How many PINs are possible if each digit must be distinct?

1. $\underset{1^{st}\#}{\underline{10}} \times \underset{2^{nd}\#}{\underline{10}} \times \underset{3^{rd}\#}{\underline{10}} \times \underset{4^{th}\#}{\underline{10}} = 10{,}000$

2. $\underset{1^{st}\#}{\underline{10}} \times \underset{2^{nd}\#}{\underline{9}} \times \underset{3^{rd}\#}{\underline{9}} \times \underset{4^{th}\#}{\underline{9}} = 7290$

3. $\underset{1^{st}\#}{\underline{10}} \times \underset{2^{nd}\#}{\underline{9}} \times \underset{3^{rd}\#}{\underline{8}} \times \underset{4^{th}\#}{\underline{7}} = \boxed{5040}$

PIN: SELECTION OF 4 DIGITS FROM 10 DIGITS, WITHOUT REPETITION, ORDER MATTERS.

$\underbrace{\text{SELECTION OF } r \text{ OBJECTS FROM } n \text{ OBJECTS WITHOUT REPETITION, ORDER MATTERS.}}$

CALLED A PERMUTATION

DEFINITION Permutation of *n* Objects Taken *r* at a Time

A permutation of a set of n distinct objects taken r at a time without repetition is an arrangement of r of the n objects in a specific order.

THEOREM 2 Number of Permutations of n Objects Taken r at a Time

The number of permutations of n distinct objects taken r at a time without repetition is given by*

$$_nP_r = n(n-1)(n-2)\cdot\cdots\cdot(n-r+1) \qquad r \text{ factors}$$

$$_5P_2 = 5\cdot 4 \text{ factors}$$

or

$$_nP_r = \frac{n!}{(n-r)!} \qquad 0 \le r \le n \qquad _5P_2 = \frac{5!}{(5-2)!} = \frac{5!}{3!}$$

Note: $_nP_n = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!} = n!$ permutations of n objects taken n at a time.

Remember, by definition, $0! = 1$.

* In place of the symbol $_nP_r$, the symbols P_r^n, $P_{n,r}$, and $P(n, r)$ are often used.

FACTORIAL: $n! = n(n-1)(n-2)\cdots(2)(1)$

$0! = 1$ BY CONVENTION

$n!$ = # WAYS TO ARRANGE (ALL) n OBJECTS.

$$\underset{1^{st}}{\underline{n}} \times \underset{2^{nd}}{\underline{n-1}} \times \underset{3^{rd}}{\underline{n-2}} \times \cdots \times \underset{n-1}{\underline{2}} \times \underset{n^{th}}{\underline{1}}$$

✓

$$_nP_r = n(n-1)\cdots(n-r+1) = \frac{n(n-1)\cdots(n-r+1)\cancel{(n-r)}\cancel{(n-r-1)}\cdots\cancel{(2)}\cancel{(1)}}{\cancel{(n-r)}\cancel{(n-r-1)}\cdots\cancel{(2)}\cancel{(1)}}$$

$$_nP_r = \frac{n!}{(n-r)!}$$

Example 2.

A musician has written and recorded 22 songs. If she want to release 10 of these songs as an album (order matters!), how many possible ways are there for her to do this?

$$\begin{array}{cccccccccc} 22 \times & 21 \times & 20 \times & 19 \times & 18 \times & 17 \times & 16 \times & 15 \times & 14 \times & 13 \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

$$= {}_{22}P_{10} = \frac{22!}{(22-10)!} = \frac{22!}{12!}$$

$$= \;\; \hookrightarrow 2.3465 \times 10^{12}$$

Example 3.

A musician has written a song with the following structure:

Verse, Chorus, Verse, Chorus, Bridge, Verse, Chorus.

Now a lyricists has written 5 different verses, 2 different choruses, and 3 different bridges. If each verse must be different, and each chorus must be the same, how many songs can be created from these lyrics with the given structure?

$$\begin{array}{ccccccc} 5 & 2 & 4 & 1 & 3 & 3 & 1 \\ \hline \text{V1} & \text{CHORUS} & \text{V2} & \text{CHORUS} & \text{BRIDGE} & \text{V3} & \text{CHORUS} \end{array}$$

(1) PICK VERSES: SELECT 3 FROM 5 ORDER MATTERS

$${}_5P_3 = 5 \cdot 4 \cdot 3 = 60$$

(2) PICK CHORUS: 2

(3) PICK BRIDGE: 3

TOTAL # POSSIBLE SONGS: $60 \times 2 \times 3 \times \boxed{360}$

DEFINITION **Combination of *n* Objects Taken *r* at a Time**

A **combination** of a set of n distinct objects taken r at a time without repetition is an r-element subset of the set of n objects. The arrangement of the elements in the subset does not matter.

ex. BEING DEALT 5 CARD POKER HAND FROM 52 CARDS.

WAYS TO DO THIS ... $_{52}C_5 = 2{,}598{,}960$

Example 4.

How many ways are there to choose 3 aces from a standard deck of 52 cards if the order doesn't matter?

3 ACES FROM 4 ACES

Let S, C, H, and D represent the ace of spades, ace of clubs, ace of diamonds, and ace of hearts, respectively.

$_4P_3 = 24$

$_3P_3 = 3!$

$_4C_3$

SCH, SHC, CSH, CHS, HSC, HCS,

CHD, CDH, HCD, HDC, DCH, DHC,

HDS, HSD, DHS, DSH, SHD, SDH,

DSC, DCS, SDC, SCD, CDS, CSD.

$\frac{24}{6} = 4$

COMBINATIONS

$$_4P_3 = 3! \times {_4C_3}$$

$$\Rightarrow \quad {_4C_3} = \frac{_4P_3}{3!} = \frac{24}{6}$$

$$= 4$$

$$_nC_r = \frac{_nP_r}{_rP_r} = \frac{_nP_r}{3!} = \frac{n!}{3!\,(n-r)!}$$

THEOREM 3 Number of Combinations of *n* Objects Taken *r* at a Time

The number of combinations of n distinct objects taken r at a time without repetition is given by*

$$_nC_r = \binom{n}{r} = \frac{_nP_r}{r!} = \frac{n!}{r!(n-r)!} \qquad 0 \le r \le n$$

$$_{52}C_5 = \binom{52}{5} = \frac{_{52}P_5}{5!} = \frac{52!}{5!(52-5)!}$$

* In place of the symbols $_nC_r$ and $\binom{n}{r}$, the symbols C_r^n, $C_{n,r}$ and $C(n, r)$ are often used.

Example 5.

If you flip a coin 20 times, you get a sequence of heads (H) and tails (T).

1. How many different sequences of heads and tails are possible?
2. How many different sequences of heads and tails have exactly 8 heads?

1. 2 2 2 2 2 2 _ _ _ _
_ _ _ _ _ _ _ _ … _ 2

$2^{20} = 1,048,576$

2. T H T T T H H H T T
T T H H T T T H T H

CHOOSE THE 8 POSITIONS, OUT OF 20, FOR THE HEADS. THE REST ARE TAILS.

(THE ORDER IN WHICH THE POSITIONS ARE CHOSEN DOES NOT MATTER)

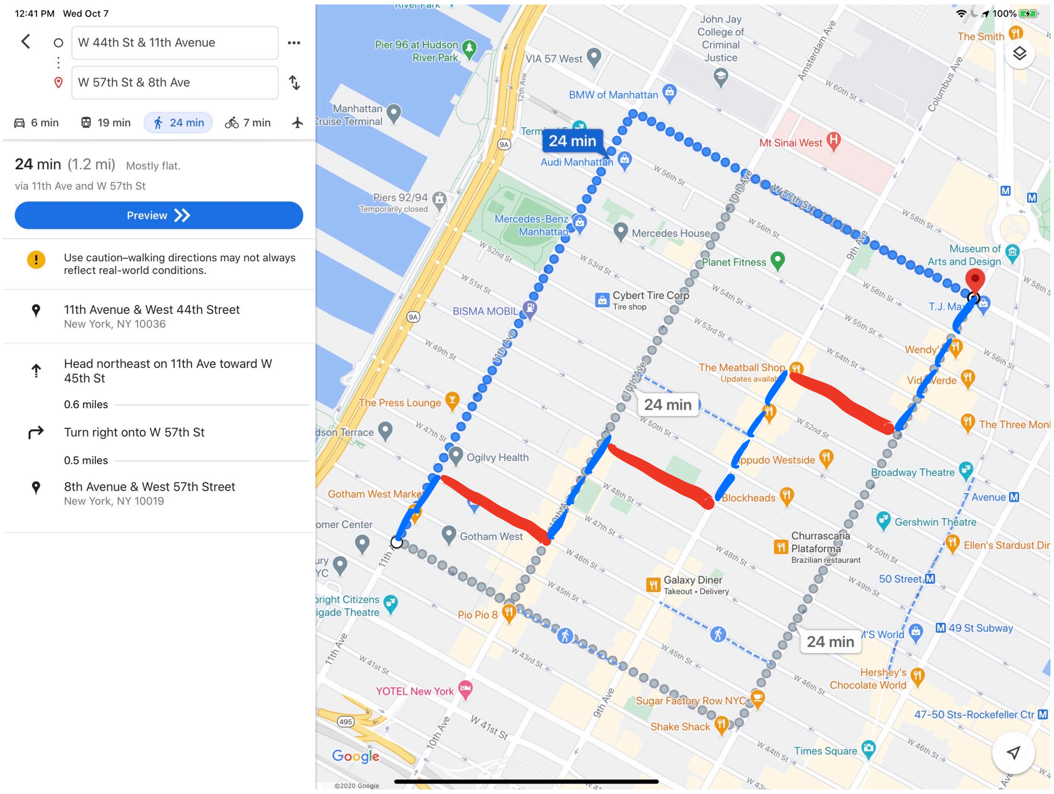
$$_{20}C_8 = \frac{20!}{8!\,(20-8)!} =$$

$$= \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot \overset{\cancel{7} \cdot 2}{\cancel{14}} \cdot 13 \cdot \cancel{12!}}{8 \cdot \cancel{7} \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{12!}}$$

$$= 19 \cdot 17 \cdot 15 \cdot 2 \cdot 13 = 125,970$$

Example 6.

How many ways are there to walk from corner of W 44 St and 11th Ave to the corner of W 57 St and 8th Ave, assuming you do not go out of your way (only walk north and east)?



NORTH 13 BLOCKS, EAST 3 BLOCKS

WALK 16 BLOCKS TOTAL.

N N E N N N E N
3 7

N N N E N N N N
12

OUT OF 16 BLOCKS CHOOSE 3 BLOCKS TO WALK EAST, ORDER DOESN'T MATTER

$$_{16}C_3 = 560$$

OUT OF 16 BLOCKS CHOOSE 13 BLOCKS TO WALK NORTH, ORDER DOESN'T MATTER

$$ {}_{16}C_{13} = 560 $$

NOTE: $$ {}_nC_r = \frac{n!}{r!(n-r)!} $$

$$ {}_nC_{(n-r)} = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!\,r!} $$

SAME

Example 7.

The fish section of a pet store is stocked with 8 guppies, 6 angelfish, 13 goldfish, and 9 rainbowfish.

1. How many ways are there to select one of each?
2. How many ways are there to select two of each?
3. How many ways are there to select 7 fish, if you must select at least one of each?

1. $\underset{\text{GUPPIES}}{\underline{8}} \times \underset{\text{ANGEL}}{\underline{6}} \times \underset{\text{GOLDFISH}}{\underline{13}} \times \underset{\text{RAINBOW}}{\underline{9}} = 5616$

2. $\underset{\text{GUPPIES}}{\underline{{}_8C_2}} \times \underset{\text{ANGEL}}{\underline{{}_6C_2}} \times \underset{\text{GOLDFISH}}{\underline{{}_{13}C_2}} \times \underset{\text{RAINBOW}}{\underline{{}_9C_2}}$

$28 \times 15 \times 78 \times 36$

$= 1,179,360$

3. (1) PICK A GUPPY - 8

(2) PICK ANGEL - 6

(3) PICK G.F. - 13

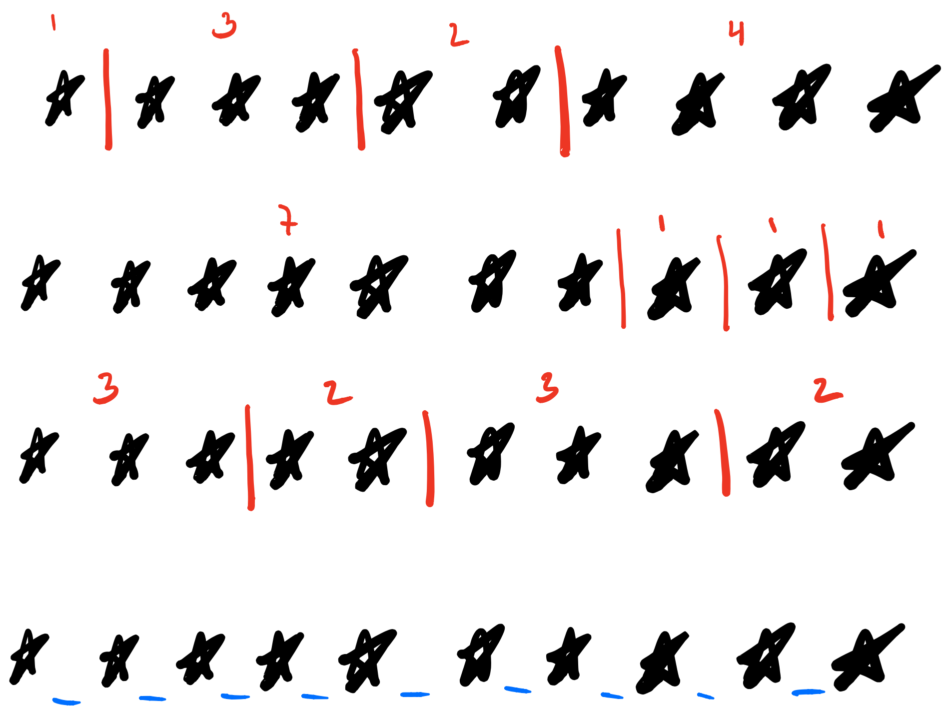
(4) PICK RF · 9

(5) PICK OTHER 3 FISH OUT OF $\quad {}_{32}C_3$

$$8 \times 6 \times 13 \times 9 \times {}_{32}C_3$$

$$5616 \times {}_{32}C_3 = 5616 \times 4960 = 27{,}855{,}360$$

Example 9.
Suppose you want to give each of your 4 nieces/nephews a Christmas card containing some money. You have ten (identical) \$20 bills that you want to distribute between the 4 (distinct) envelopes so that no envelope contains \$0. How many ways are there to do this?
https://en.wikipedia.org/wiki/Stars_and_bars_(combinatorics)



CHOOSE 3 OUT OF 9 GAPS TO PLACE BARS.

$${}_9C_3 = 84$$

2 0 8 0

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