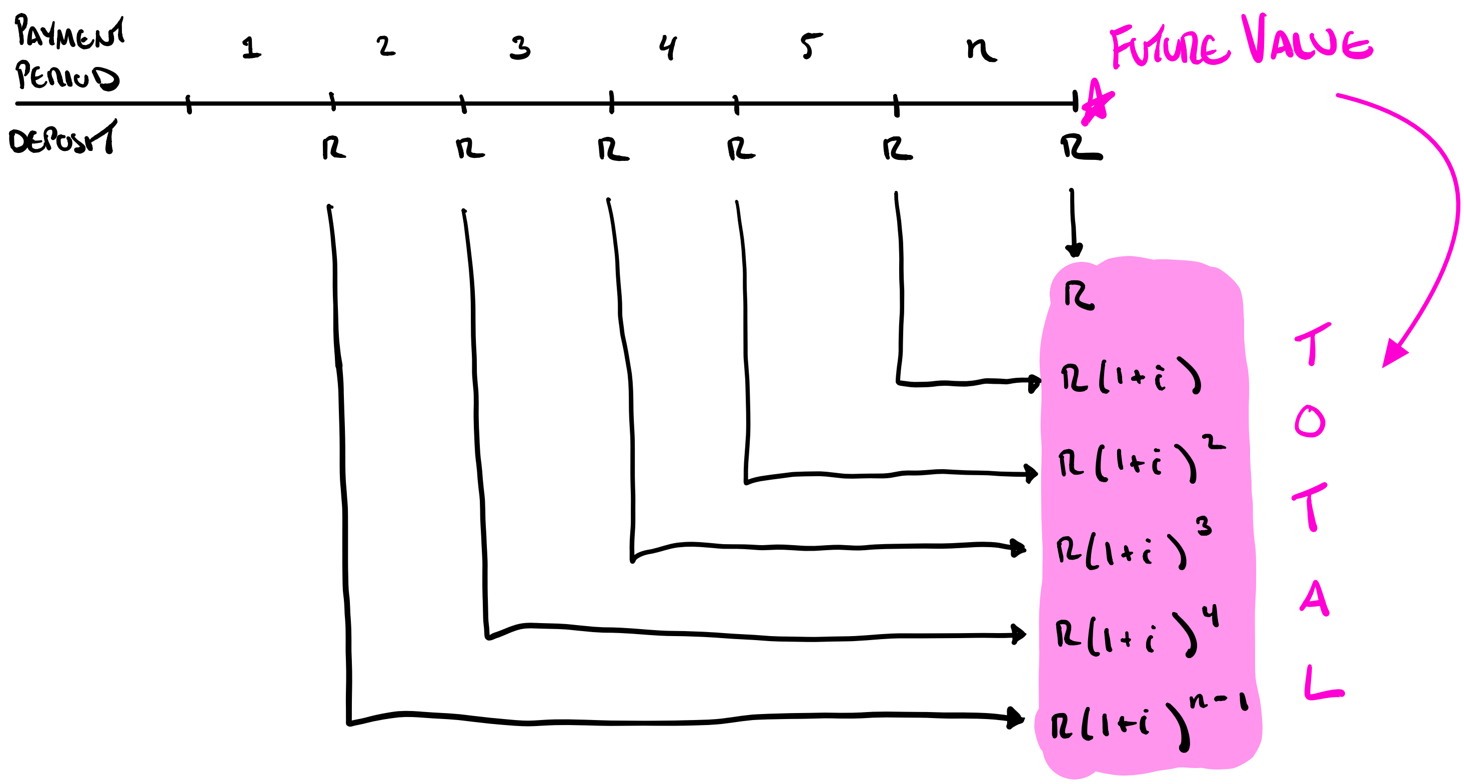
§3.3 Future Value of an Annuity

An ordinary annuity is a sequence of equal size payments R deposited at the end of each payment period into an account earning an interest rate i per payment period, compounded at the end of each payment period.

Suppose you make n payments.



Let S_n be the future value.

$$S_n = R + R(1+i) + R(1+i)^2 + \dots + R(1+i)^{n-1}$$

* Geometric series.

$$(1+i)S_n = R(1+i) + R(1+i)^2 + \dots + R(1+i)^{n-1} + R(1+i)^n$$

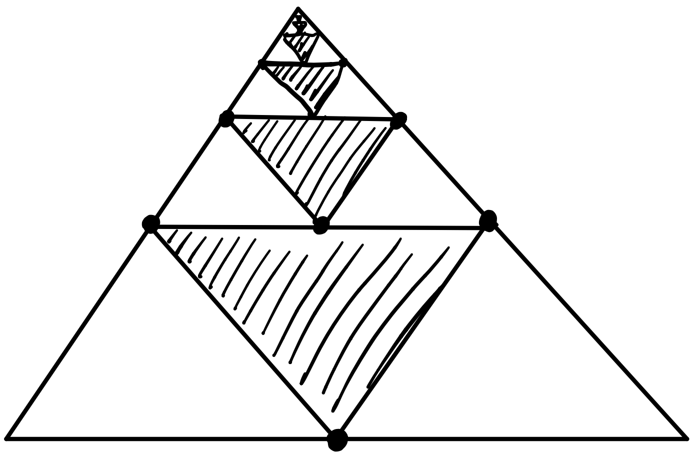
$$(1+i)S_n - S_n = R(1+i)^n - R$$

$$iS_n = R\left[(1+i)^n - 1\right]$$

$$\boxed{S_n = \frac{R\left[(1+i)^n - 1\right]}{i}}$$

Geometric Series:

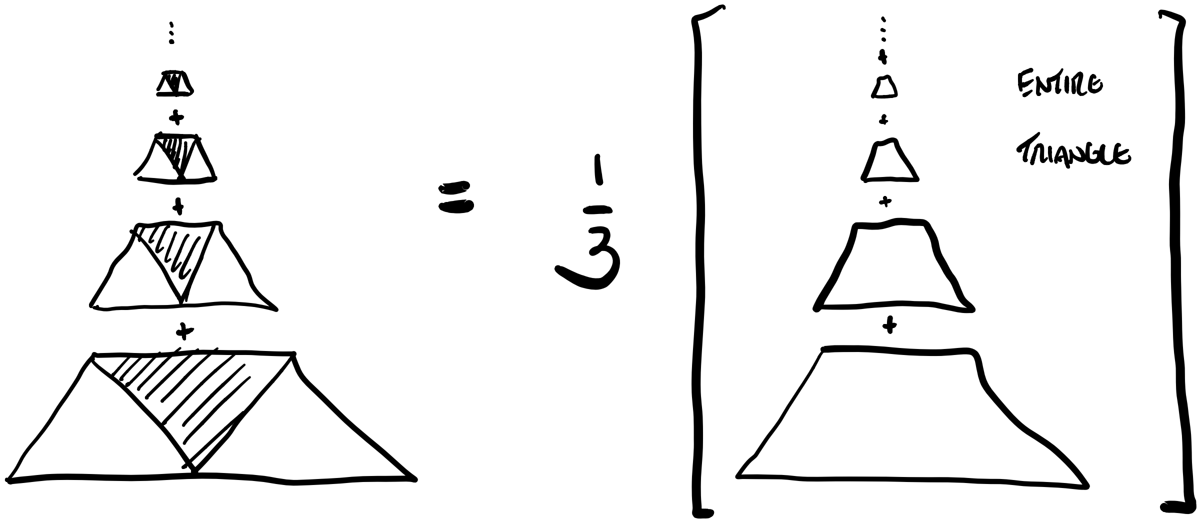
Suppose we have an equilateral triangle with area 1.



What is the area of the shaded region?

$$\underbrace{\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \dots}_{\text{sum of } \infty \text{ many terms}}$$

Geometric Series



$$= \frac{1}{3}\left[\ 1\ \right] = \frac{1}{3}$$

43. Compubank, an online banking service, offered a money market account with an APY of 1.551%.

(A) If interest is compounded monthly, what is the equivalent annual nominal rate?

(B) If you wish to have \$10,000 in this account after 4 years, what equal deposit should you make each month?

$$A = P\left(1+\frac{r}{n}\right)^{nt} = P\Big[\underbrace{\left(1+\frac{r}{n}\right)^{n}}\Big]^{t}$$

EVERY YEAR, INVESTMENT IS MULTIPLIED BY $\left(1+\frac{r}{n}\right)^{n}$

HOW MUCH BIGGER THAN 1 IS IT?

APY (ANNUAL PERCENTAGE YIELD) / EFFECTIVE RATE r_E

NOMINAL RATE

$$r_E = \left(1+\frac{r}{n}\right)^{n} - 1$$

$$r_E + 1 = \left(1+\frac{r}{n}\right)^{n}$$

$$(r_E+1)^{1/n} = 1+\frac{r}{n}$$

$$n\left[(r_E+1)^{1/n} - 1\right] = r = 12\left[(1.01551)^{1/12} - 1\right]$$

$$= .01540$$

(b.)

$$S_n = \frac{R\left[(1+i)^{n} - 1\right]}{i}$$

$S_n = 10{,}000$

$i = \frac{.01540}{12}$ INT. PER PAY. PER.

$n = $ # PAYMENTS $= 12 \cdot 4 = 48$

$R = ?$

$$R = \frac{S_n i}{(1+i)^{n} - 1} = \frac{10{,}000\left(\frac{.01540}{12}\right)}{\left(1+\frac{.01540}{12}\right)^{48} - 1}$$

$= \$202.12$

$$\text{INTEREST EARNED} = \text{FUTURE VALUE} - \overbrace{\text{TOTAL DEPOSITS}}^{nR}$$

$$= 10{,}000 - 48(202.12)$$

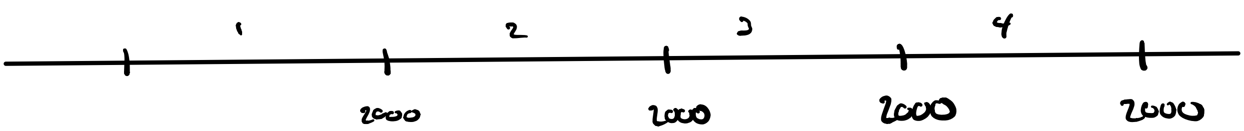
\$298.24 = 10,000 - 9,701.76

ex. IF \$2000 IS DEPOSITED EVERY 6 MONTHS INTO AN ACCOUNT EARNING 6% INTEREST COMPOUNDED SEMIANNUALLY FOR 2 YEARS, CONSTRUCT A BALANCE SHEET FOR THE INTEREST EARNED AND THE ACCOUNT BALANCE AT THE END OF EACH 6 MONTH PERIOD.

PERIOD	DEPOSIT	INTEREST	BALANCE
1	2000	-	2000
2	2000	60	4060
3	2000	121.80	6181.80
4	2000	185.45	8367.25

×.03 ×.03

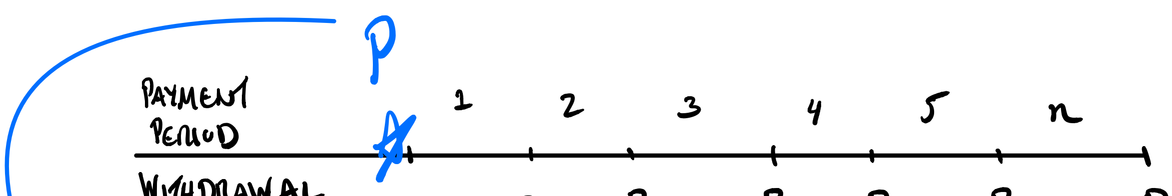
$$8367.25 = \frac{2000\left((1.03)^4 - 1\right)}{.03}$$

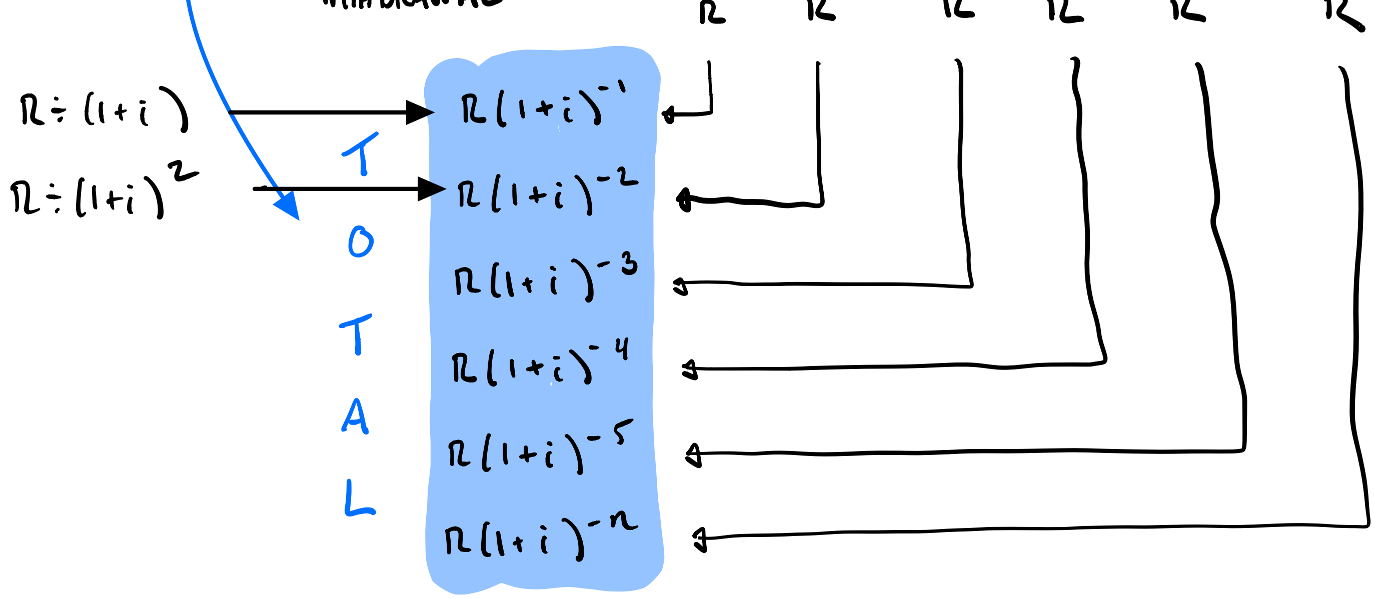


§ 3.4 PRESENT VALUE OF AN ANNUITY

NOW SUPPOSE YOU WANT TO MAKE n WITHDRAWALS OF EQUAL SIZE R AT THE END OF EACH PAYMENT PERIOD FROM AN ACCOUNT EARNING INTEREST RATE i PER PAYMENT PERIOD, COMPOUNDED AT THE END OF EACH PAYMENT PERIOD.

LET P BE THE AMOUNT OF MONEY YOU WOULD NEED TO DEPOSIT TODAY SO THAT AFTER MAKING THESE WITHDRAWALS YOUR ACCOUNT BALANCE IS \$0.





$$P = R(1+i)^{-1} + R(1+i)^{-2} + \dots + R(1+i)^{-(n-1)} + R(1+i)^{-n}$$

$$(1+i)P = R + R(1+i)^{-1} + \dots + R(1+i)^{-(n-2)} + R(1+i)^{-(n-1)}$$

$$(1+i)P - P = R - R(1+i)^{-n}$$

$$iP = R\left(1-(1+i)^{-n}\right)$$

$$\boxed{P = \frac{R\left[1-(1+i)^{-n}\right]}{i}}$$

$$\boxed{S_n = \frac{R\left[(1+i)^{n}-1\right]}{i}}$$

42. A recreational vehicle costs \$80,000. You pay 10% down and amortize the rest with equal monthly payments over a 7-year period. If you pay 9.25% compounded monthly, what is your monthly payment? How much interest will you pay?

CASH DOWN 10% of 80,000 = 8000

→ 72,000 LOAN

OUR PERSPECTIVE:

WE TAKE A LOAN OF $72,000. OUR DEBT EARNS INTEREST EVERY MONTH. WE CHIP AWAY AT OUR DEBT (+ INTEREST) WITH EQUAL SIZED PAYMENTS SO THAT AFTER 7 YEARS, DEBT IS $0.

BANK'S PERSPECTIVE:

THEY INVEST $72,000 IN YOU. THEIR INVESTMENT EARNS INTEREST EVERY MONTH. THEY MAKE EQUAL SIZE WITHDRAWALS FROM YOU, SO THAT AFTER 7 YEARS ACCOUNT BALANCE IS $0.

$$\boxed{P = \frac{R\left[1-(1+i)^{-n}\right]}{i}}$$

$P = 72{,}000$

$i = \frac{.0925}{12}$

$n = 12 \cdot 7 = 84$

$$R = \frac{Pi}{1-(1+i)^{-n}} = \frac{72000\left(\frac{.0925}{12}\right)}{1-\left(1+\frac{.0925}{12}\right)^{-84}}$$

$$= \boxed{\$1167.57}$$

$$\text{INTEREST} = \text{TOTAL PAYMENTS} - \text{INITIAL LOAN}$$
$$= nR - P$$
$$= 84(1167.57) - 72{,}000$$
$$= \$98{,}075.88 - \$72{,}000$$
$$= \boxed{\$26{,}075.88}$$

44. Construct the amortization schedule for a \$10,000 debt that is to be amortized in six equal quarterly payments at 2.6% interest per quarter on the unpaid balance.

Period	Payment	Interest	Reduction	Balance
0	-	-	-	10,000
1	1,821.58	260	1561.58	8,438.15
2	1,821.58	219.39	1602.19	6,835.96
3	1821.58	177.74	1643.84	5192.40
4	1821.58	135.00	1686.58	3505.82
5	1821.58	91.15	1730.43	1775.39
6	~~1821.58~~ 1821.55	46.16	1775.42	~~.03~~ 0 *

$$\boxed{P = \frac{R\left[1-(1+i)^{-n}\right]}{i}}$$

$$R = \frac{Pi}{1-(1+i)^{-n}} = \frac{(10{,}000)(.026)}{1 - 1.026^{-6}}$$

$$= \$1{,}821.58$$

* Last payment is adjusted ± change to make balance \$0.