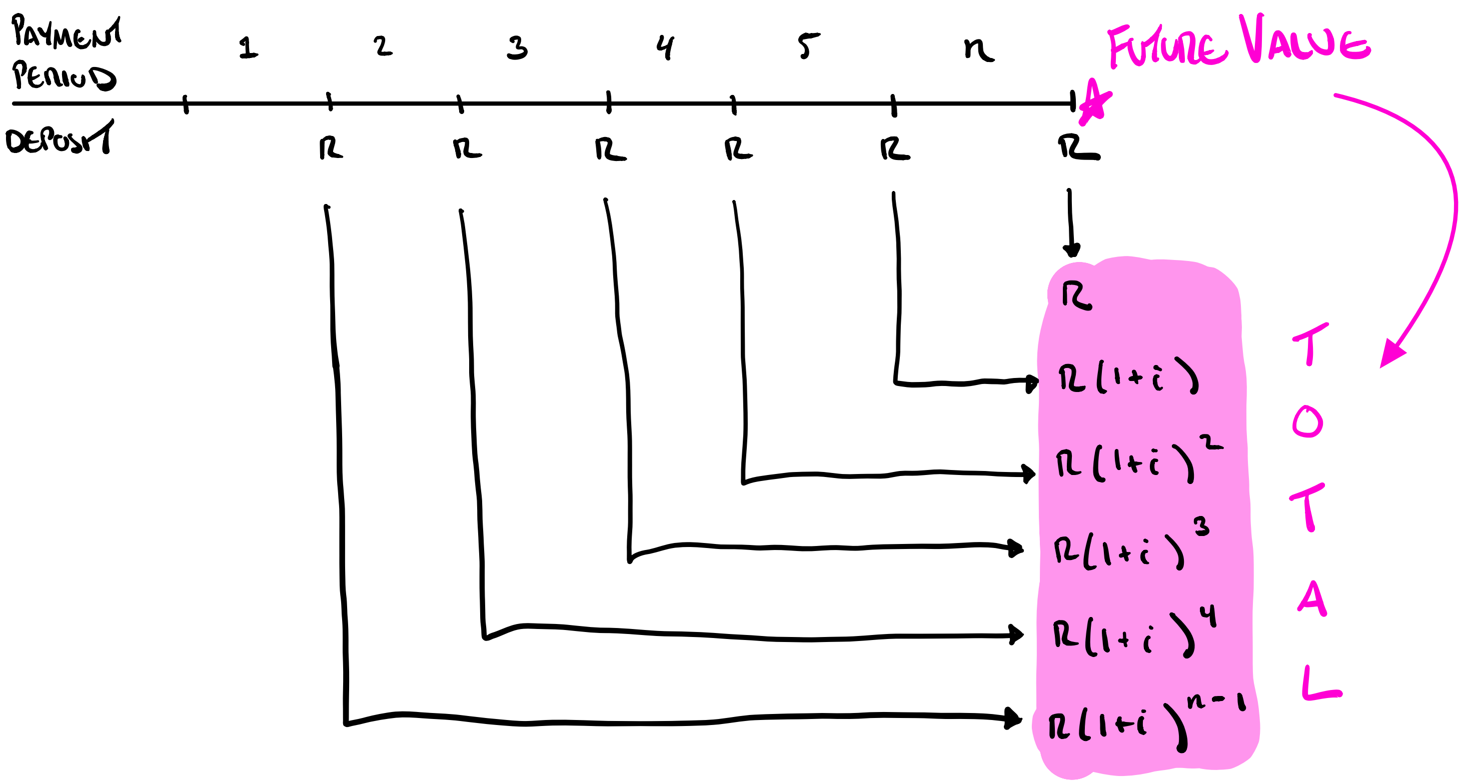
§3.3 Future Value of an Annuity

An ordinary annuity is a sequence of equal size payments R deposited at the end of each payment period into an account earning an interest rate i per payment period, compounded at the end of each payment period.

Suppose you make n payments.



Let S_n be the future value.

Geometric Series.

$$S_n = R + R(1+i) + R(1+i)^2 + \dots + R(1+i)^{n-1}$$

$$(1+i)S_n = R(1+i) + R(1+i)^2 + \dots + R(1+i)^{n-1} + R(1+i)^n$$

$$(1+i)S_n - S_n = R(1+i)^n - R$$

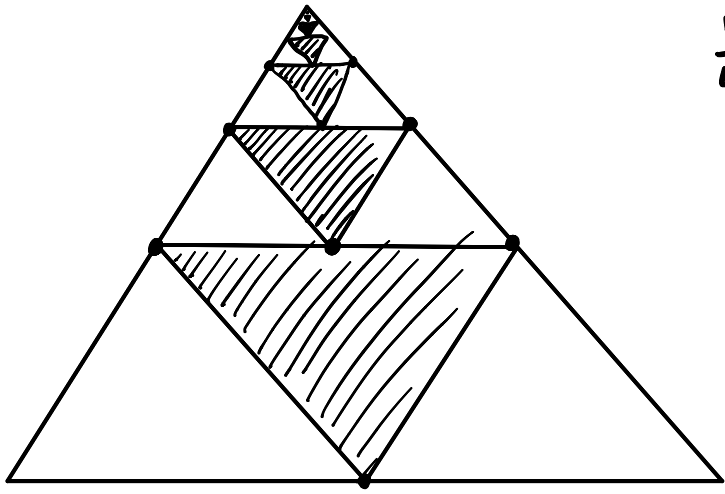
$$iS_n = R\left[(1+i)^n - 1\right]$$

$$\boxed{S_n = \frac{R\left[(1+i)^n - 1\right]}{i}}$$

Geometric Series:

Consider an equilateral triangle with area 1.

What is the area of the shaded region?

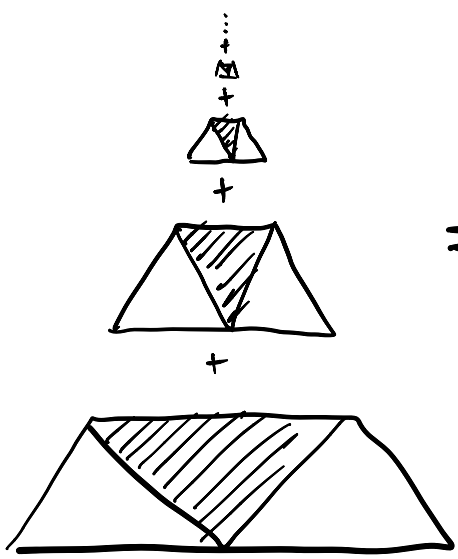


$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \dots$$

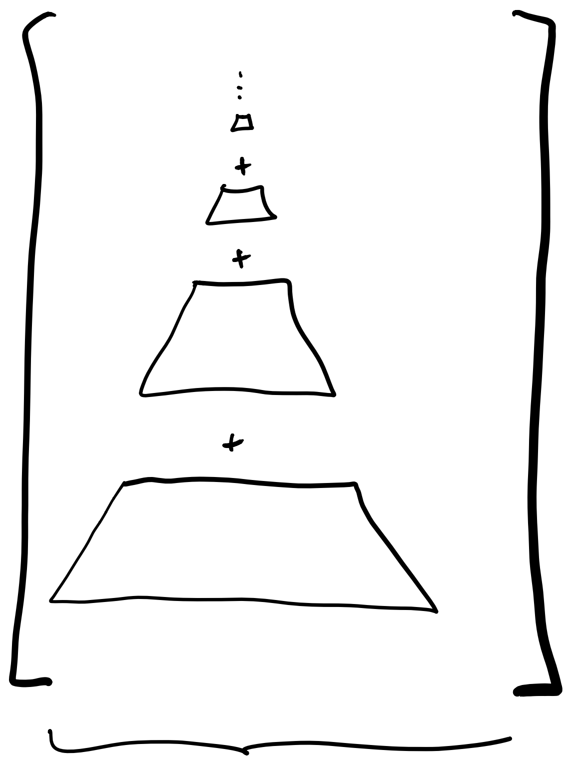
$$\dots + \left(\frac{1}{4}\right)^n$$

Infinite geometric series:

$$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots = ?$$



$$= \frac{1}{3}$$



Area 1

43. Compubank, an online banking service, offered a money market account with an APY of 1.551%.

(A) If interest is compounded monthly, what is the equivalent annual nominal rate?

(B) If you wish to have \$10,000 in this account after 4 years, what equal deposit should you make each month?

$$S_n = \frac{R\left[(1+i)^n - 1\right]}{i}$$

APY / EFFECTIVE RATE

2. Compound interest

- P = principal
- r = annual interest rate (decimal)
- n = number of compound periods per year
- t = time (years)
- A = account balance/compound amount
- r_E = effective rate/annual percentage yield (APY)

$$\boxed{A = P\left(1+\frac{r}{n}\right)^{nt} = P\left(1+r_E\right)^t} \qquad \boxed{r_E = \left(1+\frac{r}{n}\right)^n - 1}$$

(a)

$n=12$

$$r_E + 1 = \left(1+\frac{r}{n}\right)^n$$

$$\left(r_E+1\right)^{1/n} = 1 + \frac{r}{n}$$

$$n\left(\left(r_E+1\right)^{1/n} - 1\right) = r$$

$$r = 12\left[1.01551^{1/12} - 1\right] = .01540$$

(b) $$S_n = \frac{R\left[(1+i)^n - 1\right]}{i}$$

S_n = FUTURE VALUE.

R = RECURRING PAYMENT SIZE

i = INTEREST PER COMPOUND PERIOD

$= \dfrac{r}{\text{\# COMPOUNDS PER YEAR}}$

n = TOTAL # OF COMPOUND/PAYMENT PERIODS

$$R = \frac{S_n i}{(1+i)^n - 1} = \frac{10000\left(\frac{.0154}{12}\right)}{\left(1+\frac{.0154}{12}\right)^{48} - 1}$$

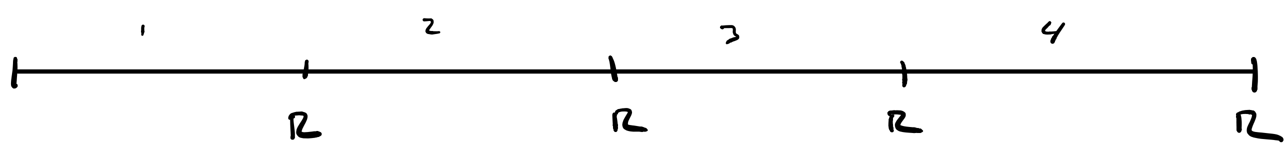
$$= \$202.12$$

TOTAL PAYMENTS $= nR = 48(202.12) = \$9701.76$

ex. IF \$2000 IS DEPOSITED EVERY 6 MONTHS INTO AN ACCOUNT EARNING 6% INTEREST COMPOUNDED SEMIANNUALLY FOR 2 YEARS, CONSTRUCT A BALANCE SHEET FOR THE INTEREST EARNED AND THE ACCOUNT BALANCE AT THE END OF EACH 6 MONTH PERIOD.

$$i = \frac{r}{\#\text{COMPOUNDS}}$$

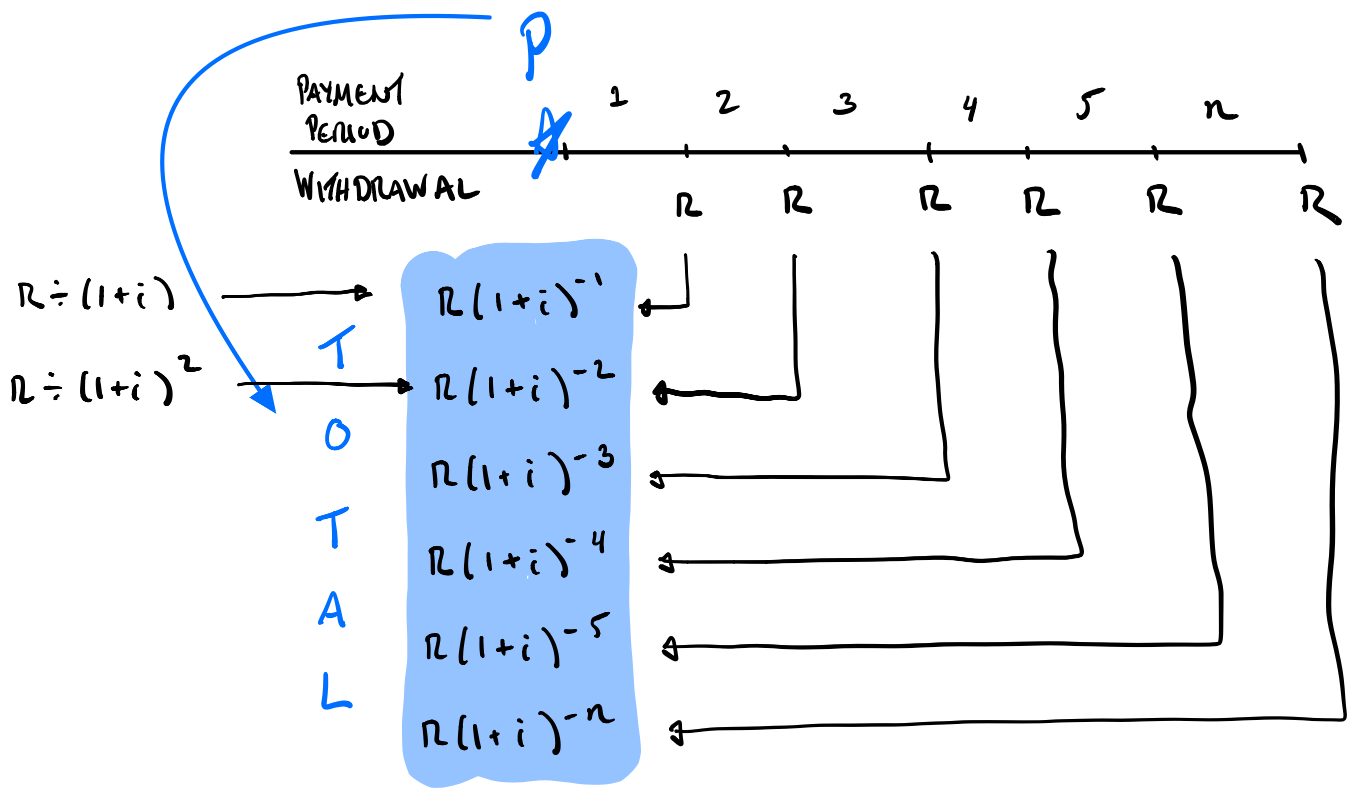
PERIOD	DEPOSIT	INTEREST	BALANCE
1	2000	– 3%	2000
2	2000	60 3%	4060
3	2000	121.80	6181.80
4	2000	185.45	8367.25



§3.4 Present Value of an Annuity

Now suppose you want to make n withdrawals of equal size R at the end of each payment period from an account earning interest rate i per payment period, compounded at the end of each payment period.

Let P be the amount of money you would need to deposit today so that after making these withdrawals your account balance is \$0.



$$P = R(1+i)^{-1} + R(1+i)^{-2} + \dots + R(1+i)^{-(n-1)} + R(1+i)^{-n}$$

$$(1+i)P = R + R(1+i)^{-1} + \dots + R(1+i)^{-(n-2)} + R(1+i)^{-(n-1)}$$

$$(1+i)P - P = R - R(1+i)^{-n}$$

$$iP = R\left(1 - (1+i)^{-n}\right)$$

$$\boxed{P = \frac{R\left[1-(1+i)^{-n}\right]}{i}}$$

$$S_n = \frac{R\left[(1+i)^n - 1\right]}{i}$$

42. A recreational vehicle costs $80,000. You pay 10% down and amortize the rest with equal monthly payments over a 7-year period. If you pay 9.25% compounded monthly, what is your monthly payment? How much interest will you pay?

10% OF 80,000 = $8,000

7 YEAR LOAN $72,000 @ 9.25% INTEREST COMPOUNDED MONTHLY.

ANNUITY FROM BANK'S PERSPECTIVE:

INVESTMENT DEPOSIT $P = 72{,}000$ INVESTED IN YOU

EVERY PAYMENT PERIOD THE BANK WITHDRAWS R FROM YOU.

AFTER FINAL WITHDRAWAL, BALANCE IS $0.

$n = 84$ $\quad i = \frac{r}{\#\text{COMPOUNDS/YR}} = \frac{.0925}{12}$

$P = 72{,}000$

$$P = \frac{R\left[1 - (1+i)^{-n}\right]}{i}$$

$$R = \frac{Pi}{\left[1-(1+i)^{-n}\right]} = \frac{(72000)\left(\frac{.0925}{12}\right)}{1 - \left(1 + \frac{.0925}{12}\right)^{-84}}$$

$$R = \$1167.57$$

INTEREST $= nR - P = 84(1167.57) - 72{,}000$

$= \$26{,}075.88$

44. Construct the amortization schedule for a $10,000 debt that is to be amortized in six equal quarterly payments at 2.6% interest per quarter on the unpaid balance.

$$P = \frac{R\left[1 - (1+i)^{-n}\right]}{i}$$

$P = 10{,}000$

$i = .026$

$n = 6$

$$R = \frac{Pi}{\left[1-(1+i)^{-n}\right]} = \frac{10000(.026)}{1 - 1.026^{-6}}$$

$R = \$1821.58$

PERIOD	PAYMENT	INTEREST	REDUCTION	BALANCE
0	-	-	-	10,000
1	1821.58	− 260	= ~~1561.58~~ 1561.58	8,438.42
2	1821.58	− 219.40	= 1602.18	6,836.24
3	1821.58	177.74	1643.84	5,192.40
4	1821.58	135.00	1686.58	3505.82
5	1821.58	91.15	1730.43	1775.39
6	~~1821.58~~ 1821.55	46.16	1775.42	~~-.03~~ 0

2.6%

2.6%

DUE TO ROUNDING TO NEAREST CENT EVERY PAYMENT PERIOD, LAST

PAYMENT MIGHT BE SLIGHTLY DIFFERENT SO THAT BALANCE GOES TO $0.