MODULE 2: LINEAR PROGRAMMING & FINDING MIN/MAX

(OPTIMIZATION)

$x, y \geq 0$

27. Minimize and maximize

$z = 10x + 30y$

subject to
$$\begin{aligned} 2x + y &\geq 16 \\ x + y &\geq 12 \\ x + 2y &\geq 14 \\ x, y &\geq 0 \end{aligned}$$

(1) GRAPH THE SOLUTION REGION

BOUNDED?

→ IF YES, THEN MIN/MAX VALUE EXIST, & OCCUR AT CORNER POINTS OF SOL'N REGION.

IF NO, THEN THERE MAY OR MAY NOT BE MIN/MAX VALUES. BUT IF EITHER EXISTS, IT OCCURS AT A CORNER POINT OF SOL'N REGION.

(2) LIST CORNER POINTS.

EVALUATE OBJECTIVE FUNCTION AT CORNER POINTS.

INTERPRET RESULTS.

THEOREM 1 Fundamental Theorem of Linear Programming

If the optimal value of the objective function in a linear programming problem exists, then that value must occur at one or more of the corner points of the feasible region.

WHERE TO LOOK FOR SOL'NS.

Theorem 1 provides a simple procedure for solving a linear programming problem, *provided that the problem has an optimal solution—not all do*. In order to use Theorem 1, we must know that the problem under consideration has an optimal solution. Theorem 2 provides some conditions that will ensure that a linear programming problem has an optimal solution.

THEOREM 2 Existence of Optimal Solutions

(A) If the feasible region for a linear programming problem is bounded, then both the maximum value and the minimum value of the objective function always exist.

(B) If the feasible region is unbounded and the coefficients of the objective function are positive, then the minimum value of the objective function exists but the maximum value does not.

(C) If the feasible region is empty (that is, there are no points that satisfy all the constraints), then both the maximum value and the minimum value of the objective function do not exist.

KNOW IN ADVANCE WHEN A SOL'N EXISTS.

TO KNOW WHEN MIN/MAX EXIST, & TO UNDERSTAND WHY THEY OCCUR AT CORNER POINTS OF THE SOLUTION REGION:

(1) CONSIDER LEVEL CURVES

(2) CONSIDER 3D GRAPHS OF OBJECTIVE FUNCTION.

LEVEL CURVES

↳ GIVEN OBJECTIVE EQ $z = Ax + By$ SET EQUAL TO CONSTANT K.

$Ax + By = K$ EQUATION WITH A GRAPH.

$$\underbrace{y = -\frac{A}{B}x + \frac{K}{B}}$$

DIFFERENT K-VALUES PRODUCE LEVEL CURVES WITH DIFF. y-INT.

A FAMILY OF **LEVEL CURVES**

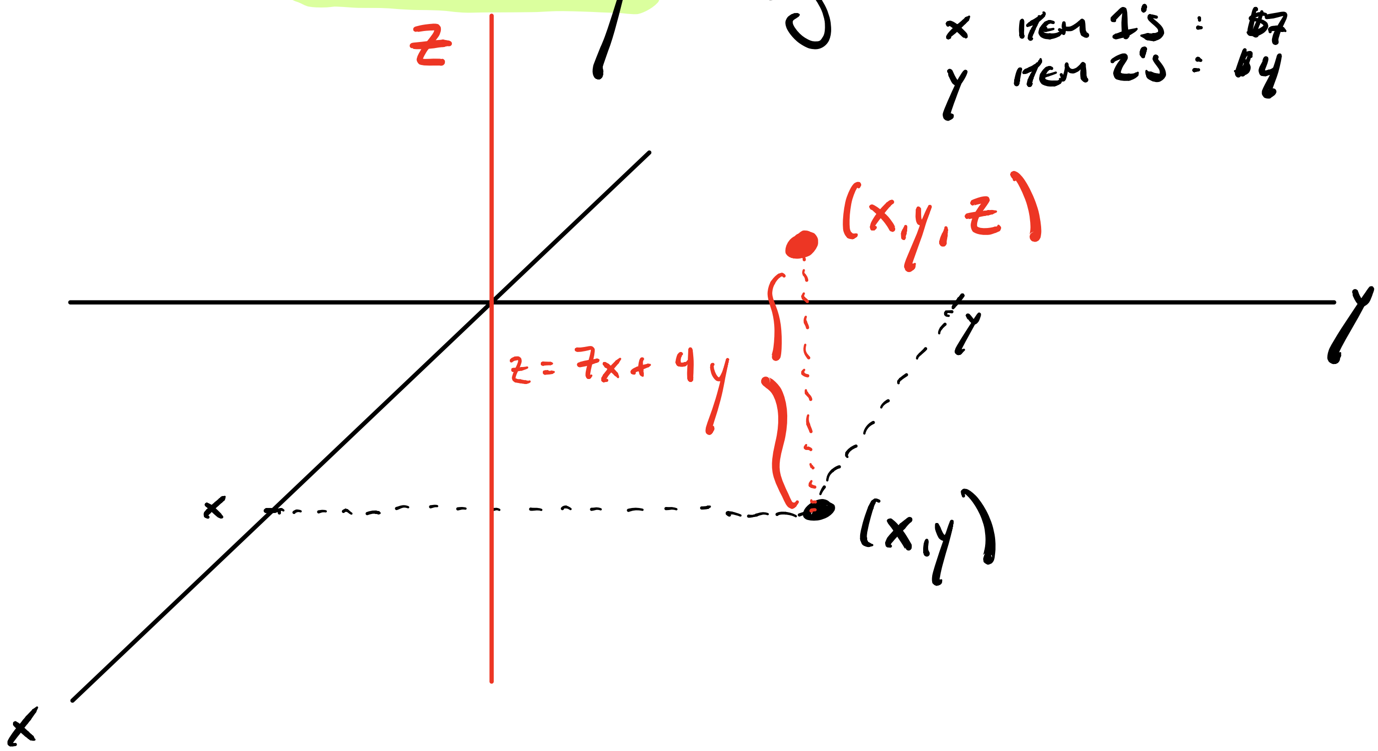
i.e. CURVES/LINES COMPOSED OF POINTS (x, y VALUES) THAT CAUSE THE OBJECTIVE FUNCTION TO EQUAL K.

THE MIN/MAX VALUES OCCUR AT BOUNDARY POINTS OF THE SOLUTION REGION WHERE LEVEL CURVES ENTER/EXIT - NECESSARILY AT CORNER POINTS.

GRAPHS OF FUNCTIONS OF 2 VARIABLES (OBJECTIVE FUNCTION)

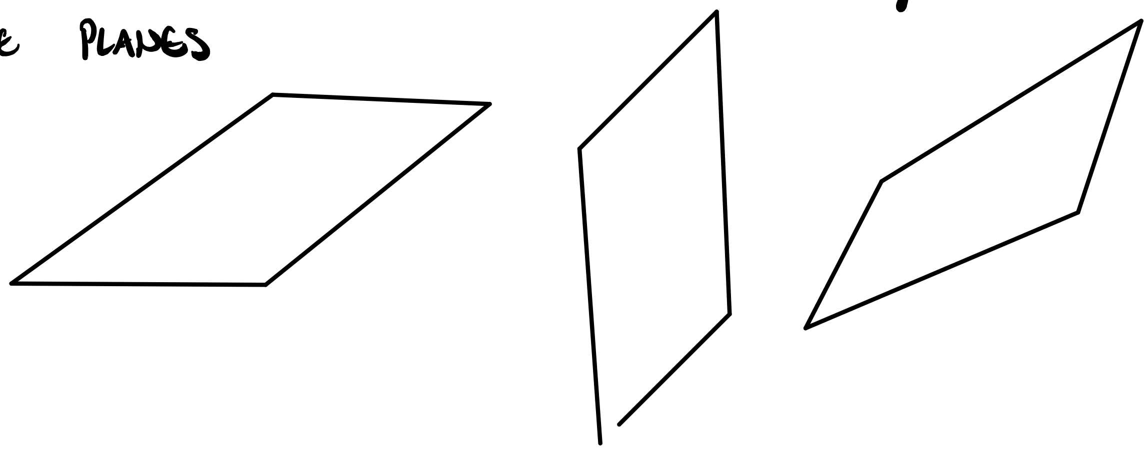
SUPPOSE $z = 7x + 4y$

e.g. COST OF PRODUCING
x ITEM 1'S : \$7
y ITEM 2'S : \$4



THE HEIGHT OF THE GRAPH ABOVE (OR BELOW) THE POINT (x, y) IS THE VALUE OF THE FUNCTION FOR THOSE VALUES OF x & y.

The graphs of linear functions are planes $z = Ax + By$ ← Linear function



Level Curves for Nonlinear Objective Functions

ex. Maximize

$$z = 3xy^2 - x^3 - y^3 - x \qquad \text{(nonlinear)}$$

Constraints: $0 \le x \le 6$, $0 \le y \le 4$.

(1) Graph sol'n region

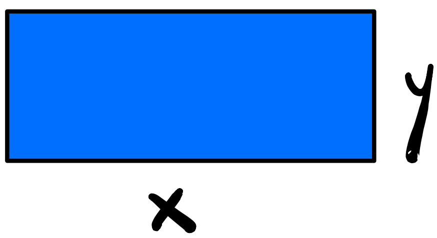
(2) Level curves: $3xy^2 - x^3 - y^3 - x = K$

Largest K such that the level curve intersects the sol'n region is the max value.

≈ 60.0206

$$\underbrace{x \approx 4, \quad y \approx 4}_{\text{(Not a corner point)}}$$

Challenge: A farmer has 100 ft of fencing & wants to enclose a rectangular region with the largest possible area. What are dimensions of the rectangle?



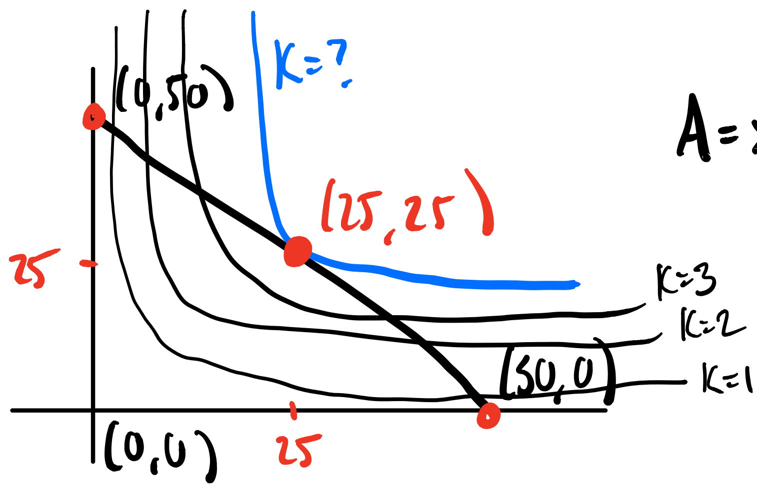
Maximize: $A = xy$

Constraints: $2x + 2y \leq 100$

$x, y \geq 0$

(1) Graph sol'n region

(2) Consider level curves $A = K \rightarrow xy = K \rightarrow y = K \cdot \frac{1}{x}$



$A = xy = 0$ at all 3 corner points

Trivial rect.

Graphically, we see max area $\approx 625 \text{ ft}^2$.

$x = y = 25$ ft