1 Bernoulli Trials and Binomial Experiments

DEFINITION Bernoulli Trials

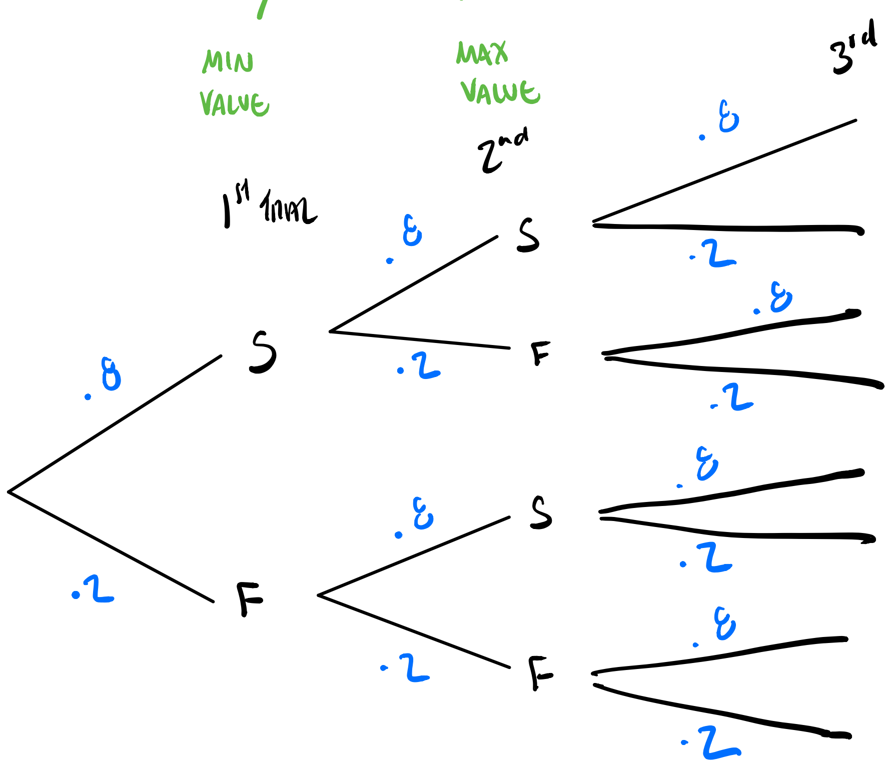
A sequence of experiments is called a **sequence of Bernoulli trials**, or a **binomial experiment**, if

1. Only two outcomes are possible in each trial.
2. The probability of success p for each trial is a constant (probability of failure is then $q = 1 - p$).
3. All trials are independent.

Definition A **binomial experiment** is one that has these five characteristics:

1. The experiment consists of n identical trials.
2. Each trial results in one of two outcomes. For lack of a better name, the one outcome is called a success, S, and the other a failure, F.
3. The probability of success on a single trial is equal to p and remains the same from trial to trial. The probability of failure is equal to $(1 - p) = q$.
4. The trials are independent.
5. We are interested in x, the number of successes observed during the n trials, for $x = 0, 1, 2, \ldots, n$.

MIN VALUE   MAX VALUE



Binomial Experiment w/ 3 Bernoulli Trials.

1. Label each of the following experiments as binomial or not binomial.

   ✗ (a) A single coin is flipped repeatedly until a head is observed and x is the number of flips.

   ✗ (b) Seven cards are dealt from a shuffled deck of 52 cards and x is the number of aces dealt.

   ✓ (c) Due to a pandemic, only 1 out of every 5 customers is allowed into a particular store. Sarah visit this store on 7 consecutive days and x is the number times she is allowed into the store.

   $n=7$, $p=1/5$, $q=4/5$

   ✗ (d) A jar contains 20 marbles: 12 red and 8 blue. Jessica selects 5 marbles from the jar simulteously and x is the number of red marbles.

   ✓ (e) A jar contains 20 marbles: 12 red and 8 blue. Jessica selects 5 marbles from the jar, replacing the marble after each selection, and x is the number of red marbles.

   $n=5$, $p=\frac{12}{20}$, $q=\frac{8}{20}$

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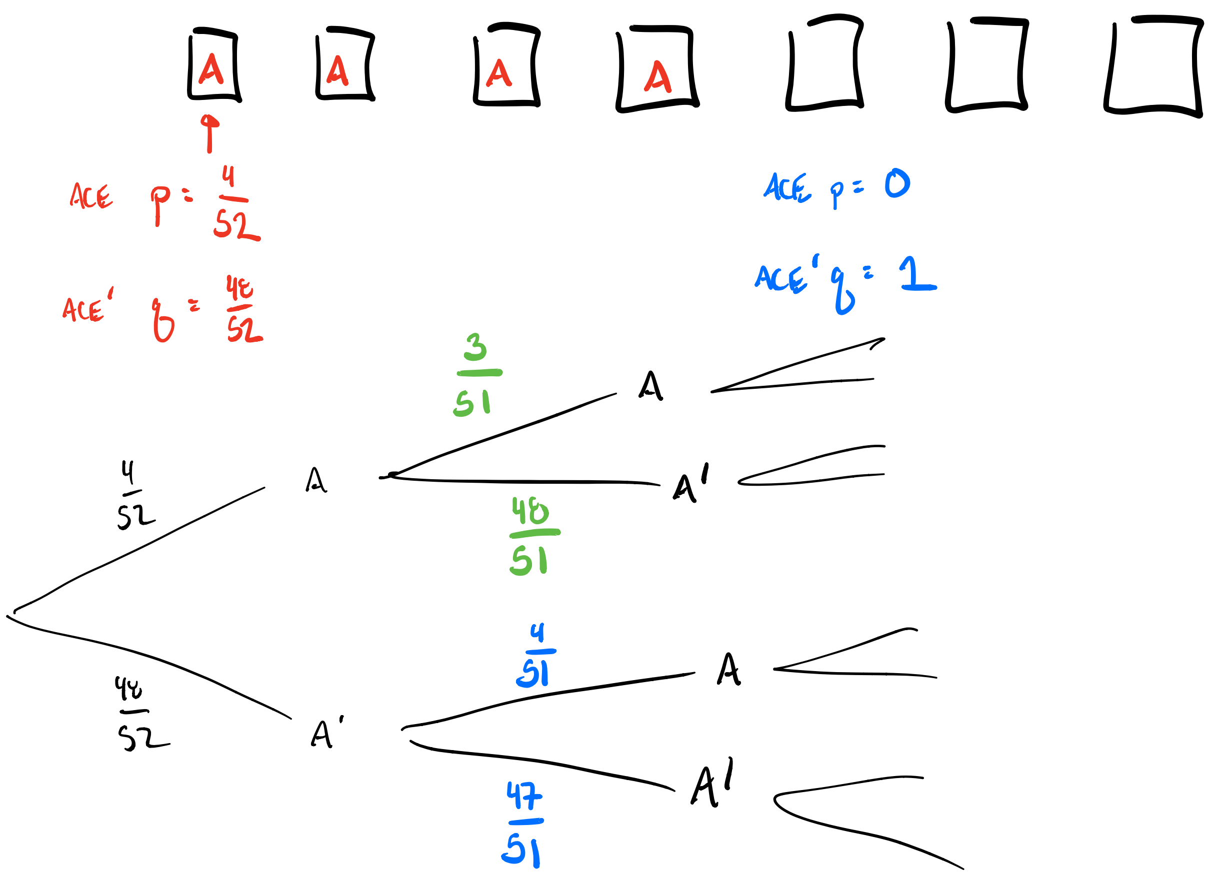
(b)



DEFINITION **Binomial Distribution**

$$P(X_n = x) = P(x \text{ successes in } n \text{ trials})$$
$$= {}_nC_x p^x q^{n-x} \qquad x \in \{0, 1, 2, \ldots, n\}$$

where p is the probability of success and q is the probability of failure on each trial. Informally, we will write $P(x)$ in place of $P(X_n = x)$.

2. Imagine two different six-sided fair dice, called die A and die B.
   - Die A has its faces labeled 1, 1, 1, 2, 2, 3.
   - Die B has its faces labeled 1, 2, 2, 3, 3, 3.

   (a) What is the probability that die A is rolled 5 times and a 2 appears exactly 3 times?

   (b) What is the probability that die B is rolled 12 times and a 1 appears exactly 3 times?

You! Try.

(a) $n = 5$ SUCCESS = A 2 APPEARS

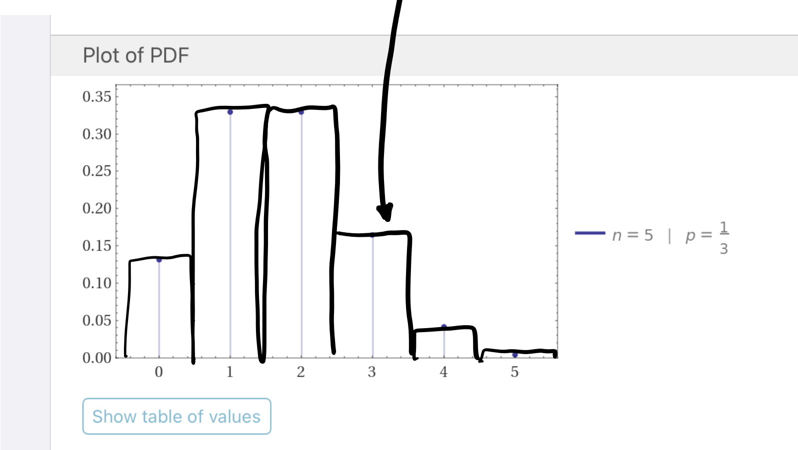
$p = \frac{2}{6} = \frac{1}{3}$

X = # SUCCESSES IN 5 ROLLS

$q = 1 - p = \frac{2}{3}$

$$P(X = 3) = {}_5C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{\overbrace{5-3}^{2}} = .1646$$

$0 \leq$ ANY INTEGER ≤ 5



(b) $n = 12$ SUCCESS = 1 APPEARS

$p = \frac{1}{6}$

$q = \frac{5}{6}$

$$P(X = 3) = {}_{12}C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^{\overbrace{12-3}^{9}}$$

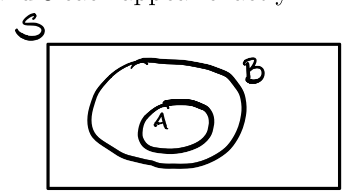
$= .1974$

3. Imagine two different six-sided fair dice, called die A and die B.

- Die A has its faces labeled 1, 1, 1, 2, 2, 3.
- Die B has its faces labeled 1, 2, 2, 3, 3, 3.

(a) What is the probability that die A is rolled 4 times and a 1 appears exactly 4 times, given that a 1 appears at least 3 times?

CHALLENGE! → (b) What is the probability that die B is rolled 6 times and the numbers 1, 2, and 3 each appear exactly twice?



(a) CONDITIONAL PROBABILITY : $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

LET A = 1 APPEARS 4 TIMES

B = 1 APPEARS AT LEAST 3 TIMES

$A \cap B = A$

FIND $P(A|B)$. WE NEED $P(A \cap B) = P(A) = {}_4C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^0$

$= \frac{1}{16} \approx .0625$

$n = 4$

$p = \frac{3}{6} = \frac{1}{2}$

$q = \frac{1}{2}$

$P(B) = P(x = 3 \text{ or } 4) = \underbrace{P(x=3)}_{{}_4C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^1 = \frac{1}{4}} + \underbrace{P(x=4)}_{\frac{1}{16}}$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{1}{16}}{\frac{1}{4} + \frac{1}{16}} = .2$

When x is the number of successes in a series of n Bernoulli trials, the mean and standard deviation for x are

$$\mu = np, \qquad \sigma = \sqrt{npq}.$$

4. Let x represent be the number of success in 20 Bernoulli trials, each with probability of success $p = .85$. Find the mean (i.e. expected value) and standard deviation for x.

ON AVERAGE, # SUCCESSES SHOULD BE 85% OF 20

$$\mu = np = (20)(.85) = 17$$

STANDARD DEV. FOR X: $\sigma = \sqrt{npq} = \sqrt{(20)(.85)(.15)}$

$$= \sqrt{2.55} \approx 1.5969$$

2 Normal Distributions

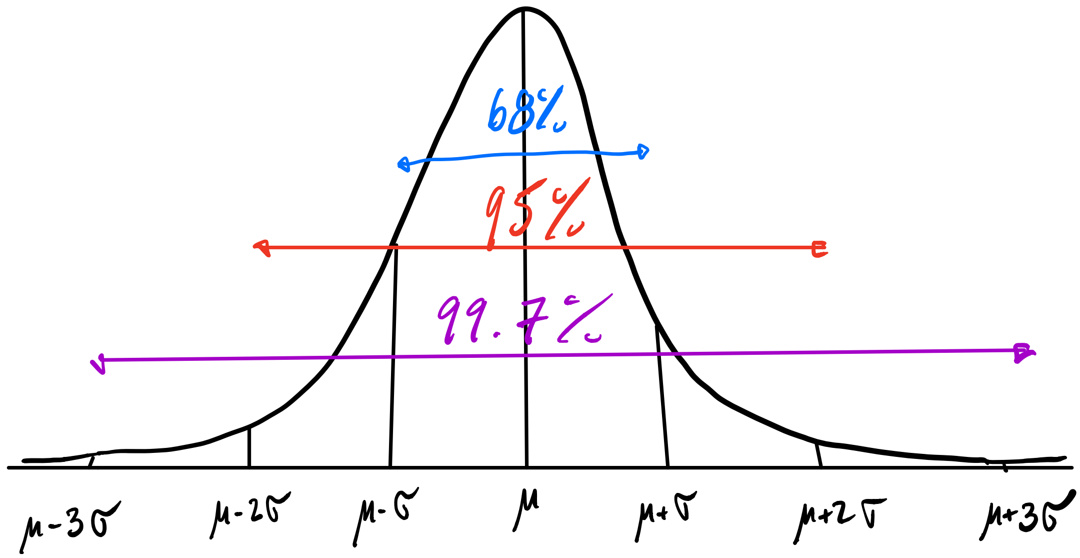


Figure 1: The 68-95-99.7 rule for normal distributions.

5. A machine in a bottling plant is set to dispense 12 oz of soda into cans. The machine is not perfect, and so every time the machine dispenses soda, the exact amount dispensed is a number x with a normal distriution. The mean and standard deviation for x are $\mu = 12$ oz and $\sigma = 0.15$ oz, respectively. Approximate the following probabilities using the 68-95-99.7 rule.

   (a) Give a range of values such that the amount of soda in 68% of all cans filled by this machine are in this range.

   (b) Give a range of values such that the amount of soda in 95% of all cans filled by this machine are in this range.

   (c) Give a range of values such that the amount of soda in 99.7% of all cans filled by this machine are in this range.

   (d) $P(11.85 \leq x \leq 12.15)$

   (e) $P(11.70 \leq x \leq 12)$

   (f) $P(x \leq 11.70)$

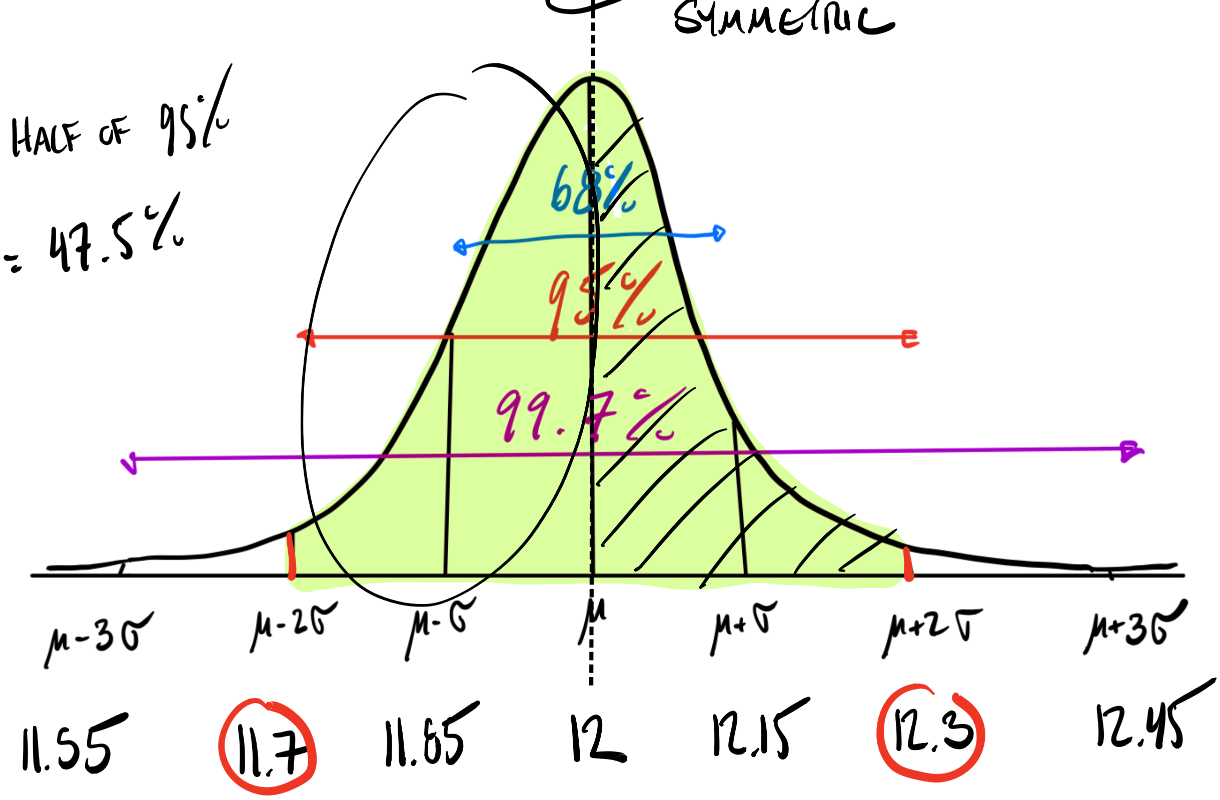
   (g) $P(12.3 \leq x \leq 12.45)$

   (h) $P(x \leq 12 \cup x \geq 12.45)$

(e)

HALF OF 95%

= 47.5%



(a) Give a range of values such that the amount of soda in 68% of all cans filled by this machine are in this range. [11.85, 12.15]

(b) Give a range of values such that the amount of soda in 95% of all cans filled by this machine are in this range. [11.7, 12.3]

(c) Give a range of values such that the amount of soda in 99.7% of all cans filled by this machine are in this range. [11.55, 12.45]

~~(d)~~ $P(11.85 \leq x \leq 12.15)$

(e) $P(11.70 \leq x \leq 12)$ = .475

(f) $P(x \leq 11.70)$

(g) $P(12.3 \leq x \leq 12.45)$

(h) $P(x \leq 12 \cup x \geq 12.45)$

Sometimes binomial distributions have the same shape as normal distriutions.

6. An experiment is composed of flipping a fair coin 100 times and counting the number of heads that appear x. Use a normal distribution and the 68-95-99.7 rule to provide rough estimates for the probabilities of the following events.
   (a) You observe between 45 and 55 heads.
   (b) You observe more than 60 heads.