§ 5.2 FUTURE VALUE OF AN ANNUITY

SEQUENCE : 5, 10, 20, 40, 80, ...

$\times r = 2$ COMMON RATIO

INITIAL (START) VALUE $a = 5$ FIRST TERM

GEOMETRIC SEQUENCE

n	1	2	3	4
n^{TH} TERM IN SEQUENCE	5	$5 \cdot 2$	$5 \cdot 2^2$	$5 \cdot 2^3$

SUMMARY: GIVEN A GEOMETRIC SEQUENCE $a, ar, ar^2, ar^3, \ldots$

THE n^{TH} TERM IN THE SEQUENCE IS $\boxed{ar^{n-1}}$.

EX. GIVEN THE SEQ : 4, 12, 36, 108, ...

FIND THE 8^{TH} TERM OF THE SEQ.

$a = 4$

$r = 3$

$n = 8$

$ar^{n-1} = 4(3)^{8-1} = 8748.$

NOW LET'S ADD UP THE FIRST n TERMS OF A GEO. SEQ.

$$S_n = a + ar + ar^2 + a^3 + \ldots + ar^{n-1}$$

* NOTE: IF $r = 1$ THEN $S_n = na$

$$- \left(S_n r = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n \right)$$

$$S_n - S_n r = a - ar^n$$

$$S_n(1 - r) = a(1 - r^n)$$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

OR $\boxed{S_n = \dfrac{a(r^n - 1)}{r - 1}} \quad (r \neq 1)$

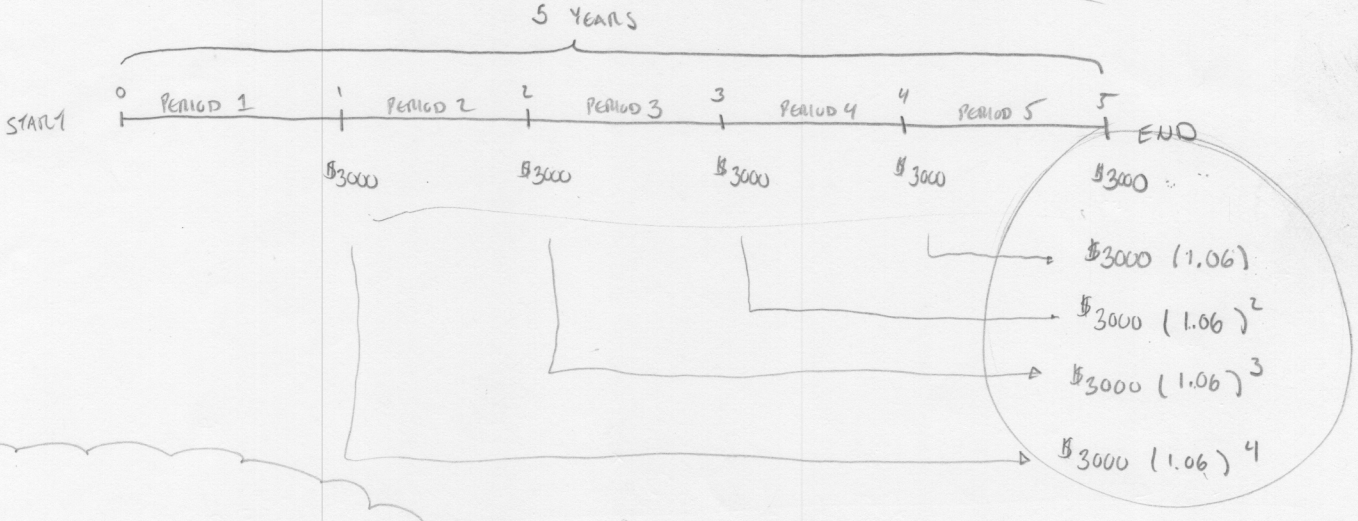
Def: AN ANNUITY IS A SEQUENCE OF EQUAL PAYMENTS MADE AT EQUAL PERIODS OF TIME.

IF THE PAYMENTS EARN COMPOUND INTEREST AND THE FREQUENCY OF COMPOUNDING IS EQUAL TO THE FREQUENCY OF THE PAYMENTS, THEN THE ANNUITY IS CALLED AN ORDINARY ANNUITY.

WE CALL THE TIME BETWEEN PAYMENTS THE PAYMENT PERIOD AND WE ASSUME THAT ALL PAYMENTS ARE MADE AT THE END OF THE PAYMENT PERIOD.

TIME FROM BEGINNING OF 1ST PAYMENT PERIOD TO THE END OF THE LAST PAYMENT PERIOD IS CALLED THE TERM OF THE ANNUITY.

ex. $3000 IS DEPOSITED AT THE END OF EACH YEAR FOR 5 YEARS IN AN ACCOUNT EARNING 6% PER YEAR COMPOUNDED ANNUALLY.



FINAL ACCOUNT BALANCE IS THE SUM OF THE COMPOUND AMOUNTS FOR ALL PAYMENTS, COMPOUNDED TO THE END OF THE TERM.

FUTURE VALUE OF THE ANNUITY.

A GEOMETRIC SUM!

$a = 3000$
$r = 1.06$
$n = 5$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

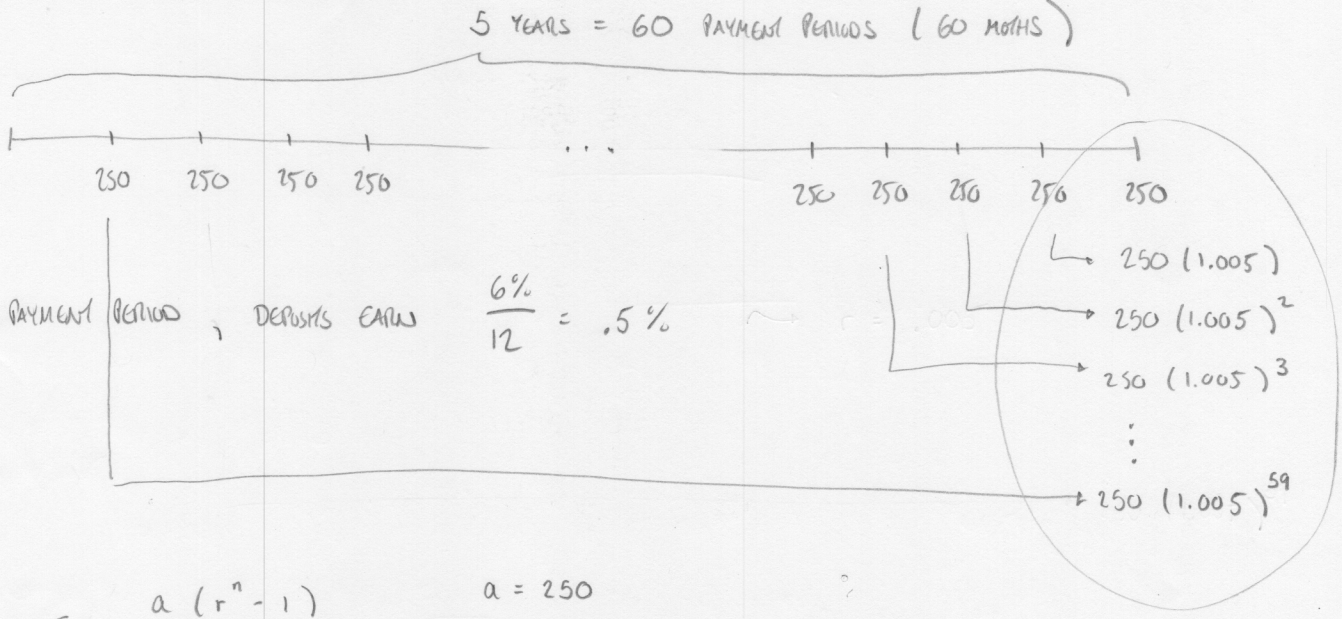
$$S_5 = \frac{3000(1.06^5 - 1)}{1.06 - 1}$$

$$= \$16911.28$$

(COMPARE TO $5(3000) = 15000$)

ex. Now suppose you deposit ₱250 every month for 5 years into an account earning 6% annual interest compounded monthly.

Find future value of annuity.



Each payment period, deposits earn $\frac{6\%}{12} = .5\%$ ⇝ $r = 1.005$

Recall: $S_n = \frac{a(r^n - 1)}{r - 1}$

$a = 250$

$r = 1.005$ interest rate per period

$n = 60$ number of time periods

Future value of annuity $= S_{60} = \frac{250(1.005^{60} - 1)}{1.005 - 1} = ₱17,442.51$

(compare to ₱16,911.28 yearly dep. ₱15,000.00 no interest)

In summary,

- a = periodic payment
- $r = 1 + \frac{\text{annual interest rate}}{\#\text{ compounds per year}}$ as decimal
- n = # payment periods = # years × # compounds (payments) per year

Future value of annuity $= S_n = \frac{a(r^n - 1)}{r - 1}$

AN ORDINARY ANNUITY WHERE PERIODIC EQUAL SIZED PAYMENTS ARE MADE IN ORDER TO HAVE A PARTICULAR FUTURE VALUE AT THE END OF THE TERM IS CALLED A SINKING FUND.

ex. SUPPOSE YOU NEED $80,000 IN 10 YEARS.

YOU WANT TO SET ASIDE A FIXED AMOUNT OF MONEY EVERY MONTH IN AN ACCOUNT EARNING 3% INTEREST COMPOUNDED MONTHLY.

HOW MUCH SHOULD YOU SET ASIDE EACH MONTH?

RECALL: $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$r = 1 + \dfrac{.03}{12} = 1.0025$

$n = 10(12) = 120$

$S_{120} = 80,000$

a = UNKNOWN

$$a = \frac{S_n(r-1)}{r^n - 1} \quad \text{(ALWAYS ROUND UP)}$$

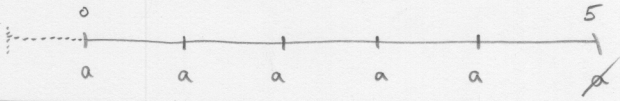
$$a = \frac{80,000(.0025)}{1.0025^{120} - 1} = 572.49$$

(NOTE: $120(572.49) = 68,698.80$ TOTAL AMOUNT DEPOSITED)

Def:

ANNUITY DUE

↳ IF PAYMENTS ARE MADE AT BEGINNING OF PAYMENT PERIODS:

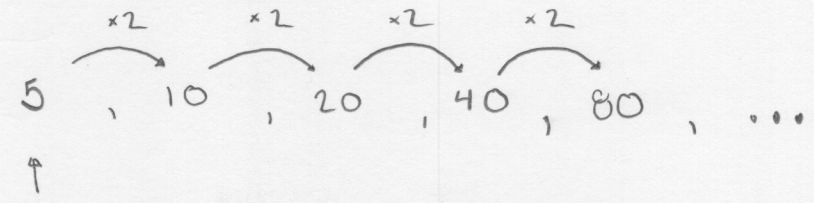


- ADD ONE TIME PERIOD $(n+1)$
- SUBTRACT LAST PAYMENT (a)

$$S_n = \frac{a(r^{n+1} - 1)}{r - 1} - a$$

§5.2 FUTURE VALUE OF AN ANNUITY.

SEQUENCE:



INITIAL VALUE b (BEGINNING)

EVERY STEP, MULT. BY q

e.g. $b = 5$, $q = 2$

GEOMETRIC SEQUENCE: $b, bq, bq^2, bq^3, \ldots$

n^{TH} TERM OF A GEOMETRIC SEQUENCE IS bq^{n-1}

n	1	2	3	...	12
n^{TH} TERM	5	10	20	...	10,240

$b = 5$

$q = 2$

bq^{1-1} bq^{2-1} bq^{3-1} bq^{12-1}

$5 \cdot 2$ $5 \cdot 2^2$ $5 \cdot 2^{11}$

NOW, LET'S ADD UP (SUM) THE FIRST n TERMS OF A GEOMETRIC SEQ. LET S_n = SUM OF THE FIRST n TERMS OF GEO. SEQ.

$$S_n = b + bq + bq^2 + \dots + bq^{n-1}$$

$$-\left[S_n q = bq + bq^2 + \dots + bq^{n-1} + bq^n\right]$$

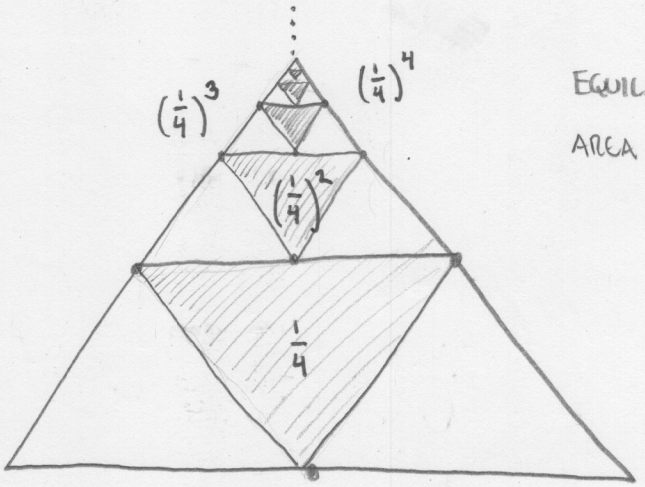
$$S_n - S_n q = b - bq^n$$

$$S_n(1-q) = b(1-q^n)$$

$$S_n = \frac{b(1-q^n)}{1-q}$$

SUM OF THE FIRST n TERMS OF A GEOMETRIC SEQUENCE

WHY GEOMETRIC?



EQUILATERAL TRIANGLE.
AREA 1.

WHAT IS THE SHADED AREA? THE TOTAL AREA OF THE FIRST n SHADED TRIANGLES IS

$$S_n = \frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\left(\frac{1}{4}\right)^2 + \dots \frac{1}{4}\left(\frac{1}{4}\right)^{n-1}$$

$$S_n = \frac{b(1-q^n)}{1-q} = \qquad b = \frac{1}{4} \qquad q = \frac{1}{4}$$

$$S_n = \frac{\frac{1}{4}\left(1 - \left(\frac{1}{4}\right)^n\right)}{1-\frac{1}{4}} \qquad \left(\frac{1}{4}\right)^n \to 0$$

$$\lim_{n\to\infty} = \frac{\frac{1}{4}(1-0)}{1-\frac{1}{4}}$$

$$= \boxed{\frac{1}{3}}$$

Account Balance is Sum of first 5 terms of Geo. Seq.

$$3000 + 3000(1.06) + 3000(1.06)^2 + \ldots + 3000(1.06)^4$$

$b = 3000$ (first term)

$r = 1.06$ (multiplied by r each term)

$$S_n = \frac{b(1-r^n)}{1-r} = \frac{3000\left(1-(1.06)^5\right)}{1-(1.06)}$$

$$S_n = \frac{b(r^n-1)}{r-1} = \frac{3000\left((1.06)^5-1\right)}{1.06-1}$$

$= \$16,911.28$

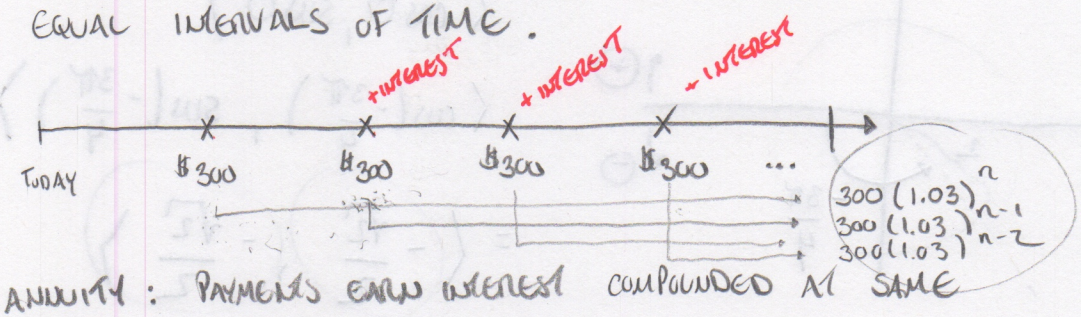
\$15,000

$I = A - \$15,000$

$\$1,911.28$

§5.2 FUTURE VALUE OF AN ANNUITY.

ANNUITY: SERIES OF EQUAL-SIZE PAYMENTS MADE AT EQUAL INTERVALS OF TIME.



ORDINARY ANNUITY: PAYMENTS EARN INTEREST COMPOUNDED AT SAME TIME INTERVALS.

FUTURE VALUE OF AN ANNUITY (ACCOUNT BALANCE AFTER ALL DEPOSITS ARE MADE), SUM OF ALL n DEPOSITS + INTEREST EARNED BY EACH

$$S_n = \frac{b(q^n - 1)}{q - 1} = \frac{R\left((1+i)^n - 1\right)}{i}$$

$S_n = b + bq + bq^2 + \ldots + bq^{n-1}$

b = RECURRING PAYMENT

q = HOW MUCH EACH PAYMENT GETS MULTIPLIED BY EACH PAYMENT PERIOD.

R = RECURRING PAYMENT

i = $\frac{\text{ANNUAL INTEREST RATE}}{\#\text{ COMPOUNDS PER YEAR}}$

n = # OF PAYMENTS.

$b = R \qquad q = 1 + i$

ex. SUPPOSE YOU DEPOSIT $250 EVERY MONTH FOR 5 YEARS INTO AN ACCOUNT THAT EARNS 6% INTEREST COMPOUNDED MONTHLY.

FIND THE FUTURE VALUE OF THE ANNUITY.

$$S_n = \frac{R\left((1+i)^n - 1\right)}{i}$$

$R = 250$

$n = 12 \cdot 5 = 60$

$i = \frac{.06}{12} = .005$

$$S_n = \frac{250\left[(1.005)^{60} - 1\right]}{.005} = \$17,442.51$$

SUMMARIZE:

SIMPLE INTEREST $I = Prt$

COMPOUND AMOUNT $A = P\left(1 + \frac{r}{n}\right)^{nt}$

P = PRINCIPAL, r = ANNUAL INTEREST RATE

t = TIME (YEARS)

n = # COMPOUNDS PER YEAR

FUTURE VALUE OF ANNUITY $S_n = \frac{R\left((1+i)^n - 1\right)}{i}$

$i = \frac{\text{INTEREST RATE}}{\text{\# COMPOUNDS/YEAR}}$

Def: AN ORDINARY ANNUITY WHERE THE PERIODIC PAYMENTS ARE MADE IN ORDER TO HAVE A PARTICULAR FUTURE VALUE IS CALLED A **SINKING FUND**.

ex. SUPPOSE YOU NEED $80,000 IN 10 YEARS.

YOU WANT TO SET ASIDE A FIXED AMOUNT EVERY MONTH INTO AN ACCOUNT EARNING 3% INTEREST COMPOUNDED MONTHLY. HOW MUCH SHOULD YOU SET ASIDE EACH MONTH?

$$S_n = \frac{R\left[(1+i)^n - 1\right]}{i}$$

$$i = \frac{.03}{12} = .0025$$

$$n = 10 \cdot 12 = 120$$

$$80{,}000 = \frac{R\left[1.0025^{120} - 1\right]}{.0025}$$

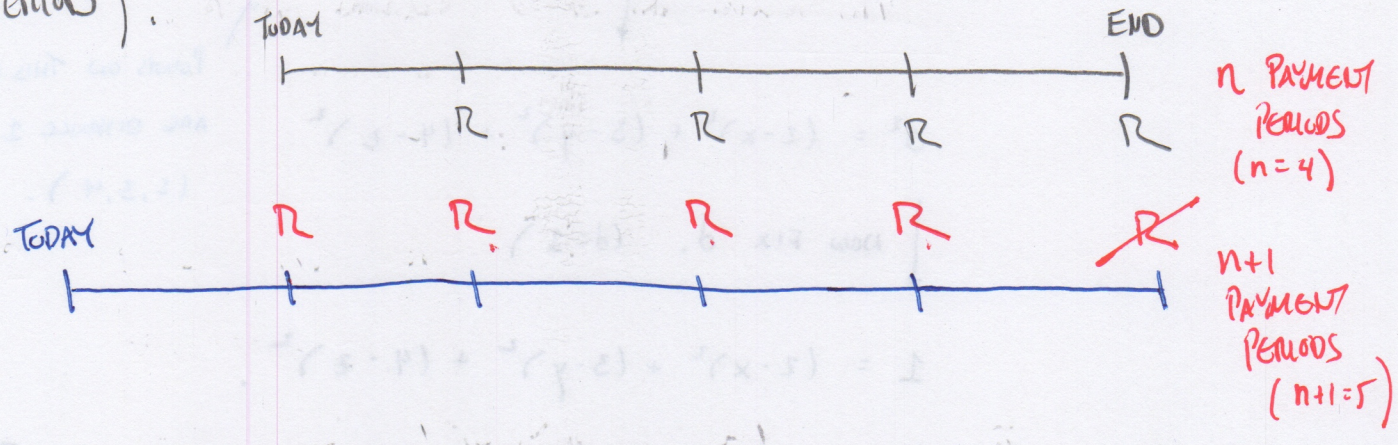
$$(80{,}000)(.0025) = R\left[1.0025^{120} - 1\right]$$

$$\frac{(80{,}000)(.0025)}{\left(1.0025^{120} - 1\right)} = R = \boxed{\$572.49}$$

(ALWAYS ROUND UP)

TOTAL AMOUNT DEPOSITED $= (\$572.49)(120) = \$68{,}698.80$

ANNUITY DUE (PAYMENTS MADE AT BEGINNING OF EACH PAYMENT PERIOD).



$$\text{ANNUITY DUE} = S_{n+1} - R$$

$$= \frac{R\left((1+i)^{n+1} - 1\right)}{i} - R$$