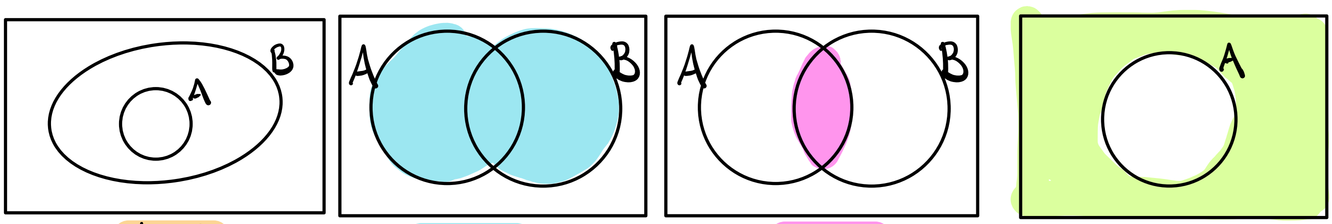
DEFINITIONS & THEOREMS TO KNOW

- A IS A SUBSET OF B ($A \subseteq B$) IF EVERY ELEMENT OF A IS AN ELEMENT OF B.
- GIVEN 2 SETS A & B, THE UNION OF A & B ($A \cup B$) IS THE SET OF ALL ELEMENTS IN A OR B (OR BOTH).
- THE INTERSECTION OF A & B ($A \cap B$) IS THE SET OF ALL ELEMENTS IN BOTH A & B.
- GIVEN A UNIVERSAL SET $\mathcal{U}$ AND A SUBSET $A \subseteq \mathcal{U}$, THE COMPLEMENT OF A (A') IS THE SET OF ELEMENTS IN $\mathcal{U}$ THAT ARE NOT IN A.



$A \subseteq B$ $A \cup B$ $A \cap B$ A'

- THE NUMBER OF ELEMENTS IN A SET A IS DENOTED $n(A)$.

ADDITION RULE: $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

1. Let U be the universal set of all New Yorkers and let
   - A be the subset of New Yorkers that know how to ride a bike,
   - B be the subset of New Yorkers that own a bike, and
   - C be the subset of New Yorkers that have gotten hurt while riding a bike.

   (a) (2 points) Use set notation (unions, intersections, compliments, etc.) to describe the set of all New Yorkers that know how to ride a bike but do not own a bike.

   (b) (2 points) In your own words, describe the set $(A \cup B)' \cap C$.

- **BASIC PROBABILITY FORMULA:**

  GIVEN A SAMPLE SPACE S OF ALL POSSIBLE OUTCOMES OF A RANDOM EXPERIMENT SUCH THAT EVERY OUTCOME IS EQUALLY LIKELY, THE PROBABILITY OF AN EVENT $A \subseteq S$ IS

$$\boxed{P(A) = \frac{n(A)}{n(S)}} \xrightarrow{\text{COROLLARY}} \boxed{P(A') = 1 - P(A)}$$

9. (a) Two people enter a room and their birthdays (ignoring the year) are recorded.
   i. What is the probability that both people are born on 1/1?
   ii. What is the probabiltiy that they have the same birthday?
   iii. What is the probability that they have different birthdays?

- **THE CONDITIONAL PROBABILITY OF A GIVEN B $\left(P(A|B)\right)$** IS

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}} \quad \text{(DEFINITION)}$$

$$\boxed{\text{PRODUCT RULE:} \quad P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)}$$

- IF ANY OF THE FOLLOWING EQUATIONS IS TRUE, THEN WE SAY **THE EVENTS A & B ARE INDEPENDENT EVENTS.** (OTHERWISE THEY ARE NOT INDEPENDENT.)

  - $P(A|B) = P(A)$
  - $P(B|A) = P(B)$
  - $P(A \cap B) = P(A)P(B)$

- TWO EVENTS A & B ARE **MUTUALLY EXCLUSIVE** IF $P(A \cap B) = 0$.

2. Suppose $P(A|B) = .8$, $P(A|B') = .4$, and $P(B) = .3$.
   (a) (6 points) Find $P(A \cap B)$.
   (b) (6 points) Find $P(A)$.
   (c) (6 points) Find $P(B|A)$.
   (d) (3 points) Are A and B independent? Why?
   (e) (3 points) Are A and B mutually exclusive? Why?

5. If A and B are events such that $P(A) = 0.5$ and $P(A' \cap B') = 0.3$, find $P(B)$ when:
   (a) A and B are mutually exclusive;
   (b) A and B are independent.

8. A 2012 Pew Research survey collected data on 2,373 randomly sampled registered voters. The results were as follows: 35% of respondents identified as Independent, 23% identified as swing voters, and 11% identified as both.
   (a) Are "being Independent" and "being a swing voter" mutually exclusive?
   (b) Draw a Venn diagram summarizing the variables and their associated probabilities.
   (c) What percentage of voters are Independent but not swing voters?
   (d) What percentage of voters are Independent or swing voters?
   (e) What percentage of voters are neither Independent nor swing voters?
   (f) Are the events "Independent" and "swing voter" independent?

THE LAW OF TOTAL PROBABILITY:

GIVEN A COLLECTION OF EVENTS $S_1, S_2, \ldots, S_K \subseteq S$
SUCH THAT

(1) $S_1 \cup S_2 \cup \ldots \cup S_K = S$

(2) $S_i \cap S_j = \phi$ FOR ALL $i \neq j$

WE HAVE

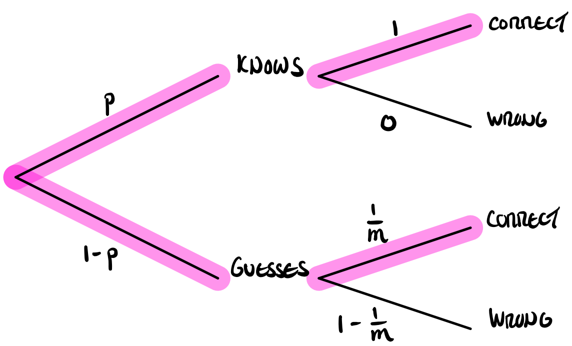
$$P(A) = P(S_1 \cap A) + P(S_2 \cap A) + \ldots + P(S_K \cap A)$$

$$\Rightarrow \boxed{P(A) = P(S_1)P(A|S_1) + P(S_2)P(A|S_2) + \ldots + P(S_K)P(A|S_K)}$$

BAYES' THM: $$\boxed{P(B|A) = \frac{P(B)P(A|B)}{P(A)}}$$

USUALLY LAW OF TOTAL PROBABILITY IS REQUIRED HERE!

Example 2. In answering a question on a multiple-choice test, a student either knows the answer or guesses. Let p be the probability that the student knows the answer and $1-p$ be the probability that the student guesses. Assume that a student who guesses at the answer will be correct with probability $1/m$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that he or she answered it correctly?



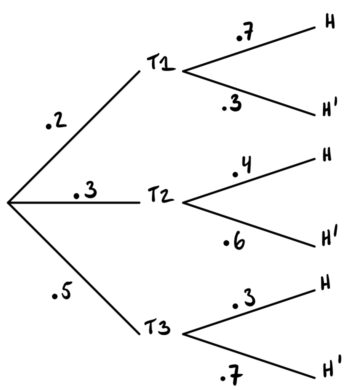
$$P(\text{KNOWS} \mid \text{CORRECT}) = \frac{P(\text{KNOWS})P(\text{CORRECT} \mid \text{KNOWS})}{P(\text{CORRECT})}$$

$$= \frac{P(\text{KNOWS})P(\text{CORRECT} \mid \text{KNOWS})}{P(\text{KNOWS})P(\text{CORRECT} \mid \text{KNOWS}) + P(\text{GUESSES})P(\text{CORRECT} \mid \text{GUESSES})}$$

$$= \boxed{\frac{p}{p + (1-p)\frac{1}{m}}}$$

Example 7. A bin contains 3 different types of disposable flashlights. The probability that a type 1 flashlight will give over 100 hours of use is .7, with the corresponding probabilities for type 2 and type 3 flashlights being .4 and .3, respectively. Suppose that 20% of the flashlights in the bin are type 1, 30% are type 2, and 50% are type 3.

1. What is the probability that a randomly chosen flashlight will give more than 100 hours of use?
2. Given that a flashlight lasted over 100 hours, what is the conditional probability that it was a type j flashlight, $j = 1, 2, 3$?



(a) $P(H) = P(T_1)P(H|T_1) + P(T_2)P(H|T_2) + P(T_3)P(H|T_3)$
$= (.2)(.7) + (.3)(.4) + (.5)(.3)$
$= .41$

(b) $P(T_1 \mid H) = \frac{P(T_1)P(H|T_1)}{P(H)} = \frac{(.2)(.7)}{.41} = .3415$

$P(T_2 \mid H) = \frac{P(T_2)P(H|T_2)}{P(H)} = \frac{(.3)(.4)}{.41} = .2927$

$P(T_3 \mid H) = \frac{P(T_3)P(H|T_3)}{P(H)} = \frac{(.5)(.3)}{.41} = .3659$

MORE CHALLENGING (HARDER THAN EXAM QUESTIONS):

Example 9. A simplified model for the movement of the price of a stock supposes that on each day the stock's price either moves up 1 unit with probability p or moves down 1 unit with probability $1 - p$. The changes on different days are assumed to be independent.

1. What is the probability that after 2 days the stock will be at its original price?
2. What is the probability that after 3 days the stock's price will have increased by 1 unit?
3. Given that after 3 days the stock's price has increased by 1 unit, what is the probability that it went up on the first day?

Example 10. A true–false question is to be posed to a husband-and-wife team on a quiz show. Both the husband and the wife will independently give the correct answer with probability p. Which of the following is a better strategy for the couple?

1. Choose one of them and let that person answer the question.
2. Have them both consider the question, and then either give the common answer if they agree or, if they disagree, flip a coin to determine which answer to give.